GETTING AT THE TRUTH

Notes for a public address given in Halifax, NS on Monday, June 14 by Edward J. Barbeau, University of Toronto, Toronto, ON M5S 3G3 (*barbeau@math.utoronto.ca*). Only some of the problems were discussed in the talk.

Mathematicians are widely regarded as people who calculate things. Even those more familiar with the discipline, such as engineers or economists, may see the subject as a fixed body of knowledge and technique that can be applied to the solutions of certain types of problems. Indeed, this ethos permeated much of school mathematics in the past and modern reformers of the curriculum have tried to broaden this view to an appreciation of the creativity and eclecticism of the discipline.

A great deal of the popularization of mathematics (and there is a lot of excellent material available) focusses on the unexpectedly wide range of application, on some of its main ideas or on particular problems. It is difficult to convey the essence of a subject that can be highly technical and complex in its methodology, discourse and application to those whose experience has been shaped by standard school work and the mostly arithmetical manifestations in ordinary life. My goal today is to use a sample of problems to to indicate in a straightforward way some of the analysis that is the life-blood of mathematics. Indeed, mathematical reasoning is often no more complex that that used by people in other situations, particularly in playing games of strategy such as checkers or bridge. The fact that such enterprises are common in the general population should encourage us in the belief that we can in the school curriculum introduce material that more authentically reflects the activities of mathematicians. I hope that some of the audience have had a chance to think about those problems I circulated in advance of this talk. But, no matter. It should be possible for someone coming at them fresh to appreciate the points I wish to make about mathematical process.

First, let me define the plane of discussion. While solving problems is a central activity of mathematicians, it is too narrow to define them simply as problem solvers. Mathematicians are really after understanding. Every problem arises in a context, and it is important to become familiar with the context and grasp the confluence of structures, facts and intuitions that make the problem intelligible and places its solution (if it can be found) in a broader universe. Our questions are really not so much, "What is the answer?" but "What is going on? How do things fit together? Where does this lead?"

Mathematicians are sculptors of ideas. A great deal of the research literature is devoted, not to discovering and establishing new results, but of taking existing results, reworking their proofs, looking at them from different angles and searching for structures that reveal their significance and serve as a basis for generalization. It is almost a decade since Andrew Wiles achieved fame for his solution of the Fermat problem. But he did much more than just solve an individual problem; he placed the problem in a setting that underscored its significance to related lines of enquiry in a way whose ramifications are still being worked out.

Everyone who does mathematics above the most trivial level learns that often the key is to be patient. "Live with" a problem situation for a while; turn it around in your mind, and perhaps a new perspective will come that allows you to come to grips with it. Let us look at some examples.

I would like to make a comment on the exposition below. In print, the explanations look quite formidable because I have tried to give a full discussion of the solutions. Many readers may feel that it makes an undue demand on their powers of concentration. This may be to some extent my own fault, but it also reflects the fact that doing mathematics demands the ability to follow a sustained line of systematic analysis. However, I have done many of these problems with school students and lay people, and found that in the process of discussion, the issues involve tend to come out more clearly and understanding results.

1. **The clipped chessboard.**

You have a regular 8×8 chessboard with alternating black and white squares. You also have a supply of dominoes, each of which can cover exactly two adjacent squares of the chessboard. It is easy to see how you can cover the chessboard with 32 non-overlapping dominoes.

Now suppose that you clip off from the chessboard two diagonally opposite corner squares, so that you are now left with 62 squares. Try to cover the chessboard with 31 non-overlapping dominoes.

One's first inclination is just to try to perform the task in some way. One way to get inside is to look at a smaller problem of the same time. The nature of the problem requires that we deal with chessboards with an even number of squares. The job cannot be done with a 2×2 chessboard, but this seems a rather special situation. With a 4×4 chessboard, a trial-and-error approach is manageable. We see that at least one of the dominoes adjacent to a clipped corner square must lie along an edge. Once we place this domino, then the rest essentially place themselves in a sequence and we can see that the task is not possible in this case. We have not solved the problem as stated, but we might try to operate on the belief that the 4×4 case might be indicative of what happens more generally. Moving on to the 6×6 chessboard, we feel that we might barge our way to a similar conclusion, but now the task is beginning to look too much like work. Is there a way we get a handle on the situation that will avoid this?

As in many mathematical situations, one has to get out of a particular rut and make a key observation. Here, we need to note that, with the chessboard coloured in the standard way with black and white squares, *each domino must cover exactly one white and exactly one black square*, so that no matter how we place them on the board, altogether an equal number of black and of white squares are covered. But the diagonally opposite corner squares of a chessboard have the same colour, so the other colour predominates in what is left over. The covering is not possible.

This argument not only avoids the tedious trial and error, but gives us a bonus. The actual size of the chessboard is not important; the same argument works whenever we have a chessboard with *any* even number of squares to a side. This situation illustrates something that often happens in mathematical research. A problem, whose solution seems to lead to greater and greater complications to the degree than it seems uninteresting and hardly worth bothering about, is redeemed by an insight into its structure that results in progress that seemed previously impossible.

2. **A deck of cards.**

Suppose that you have an ordinary deck of 52 playing cards, 26 of which are red (hearts and diamonds) and 26 black (spades and clubs). Split it into two piles, with all the reds in one pile and all the blacks in the other. Take any seven cards from the black pile and shuffle them thoroughly into the red pile, to get a pile of 33 cards, red interspersed with some black. Take any seven cards from the larger pile and put them with the black pile. You end up with two piles of 26 cards, only this time it is likely that each pile will contain cards of both colours.

Here is the question: will there be more black cards in the pile that was originally all red than there are red cards in the pile that was originally all black?

There is a more popular version of this problem. You have a quart of wine and a quart of water. A cup of wine is transferred to the water vessel and mixed in thoroughly; then a cup of the mixture is transferred to the wine vessel. Is there more wine in the water vessel than water in the wine vessel. This sort of problem often favours the person who has *not* had much mathematics. One's very experience may lead one to define a variable and attempt an algebraic solution. With sufficient care, this can be successful and lead to an answer that may turn out to be unexpected, and perhaps make you wonder if there is something that you should have realized in the first place.

Once again, one might miss the forest for the trees (the technical algebra). The key observation is that both piles of cards (or vessels in the reformulation) have the same capacity, so that what is missing from one winds up in the other. The black cards that wind up in the red pile have displaced exactly the same number of red cards which must wind up in the black pile. The water that displaces the wine must have sent exactly that amount of wine to the water vessel.

Note that this argument does not involve technical mathematical background, although the sort of mathematical experience that encourages a broader structural approach will help.

3. **The pyramid and the tetrahedron.**

You have a square-based pyramid and a regular tetrahedron (a triangular-based tetrahedron); all the edges of both solid figures have length equal to 1. The pyramid has five sides, one of which (its base) is a square and four of which are equilateral triangles. All four sides of the tetrahedron are equilateral triangles.

The two solid figures are glued together so that one of the triangular faces of the tetrahedron exactly covers one of the triangular faces of the pyramid. How many faces does the resulting solid have?

This seems like a pretty easy question. The pyramid has five faces and the tetrahedron four. However, one of the faces of the pyramid is hidden by one of the faces of the tetrahedron, so that there are four faces of the pyramid and three of the tetrahedron in view. The total number of faces is seven.

This question was actually posed on an American aptitude examination with "seven" as the expected answer. However, an objection was raised – the answer should have been "five". It turns out that two of the faces of the pyramid end up along the same planes as two faces of the tetrahedron. This is not so easy to appreciate if you think of the tetrahedron as being some kind of carbuncle adhering to a face of the pyramid. It is natural to conceive of a tetrahedron as resting on a triangular face, with the other three faces rising to a point. After all, this is what it looks like when it is resting on a table.

But there is another way of conceiving of a tetrahedron. Each pair of opposite edges are skew lines (non-parallel and non-intersecting in space) that point in perpendicular directions. So we can imagine our tetrahedron as resting (precariously, if you like) on one edge. To appreciate how it lies with the pyramid, imagine that we put two pyramids side by side on the table, abutting along an edge of the base. Each pyramid is in the position of the other shifted over by 1 unit, so their apexes are distance 1 apart. The common edge of the pyramids and the segment joining the apexes are two skew lines unit distance apart, and their endpoints are unit distance away from each other. In short, it is possible to fit the tetrahedron in the space between the two pyramids, and it now becomes clear that the faces of the tetrahedron that are visible are continuations of faces of the pyramids. (I recommend that you make models of two pyramids and one tetrahedron and try it.)

I would like to move to two problems that involve a transformation of viewpoint of a sort that often appears to clarify a mathematical analysis.

4. **Sam's journey.**

Suppose Sam lives in Halifax and has an aunt who lives in Truro. One day at 9 am, he set out on his bike to visit the aunt, and arrived at her house at 4 pm. The following day, starting at 9 am, he returned home on his bike, taking exactly the same route. Of course, on both days, he travelled at varying speeds, sometimes stopping for a rest and sometimes speeding down the hills.

Every point that Sam passed on the way out, he passed again on the way home. Was there any point that he passed at exactly the same time on both days?

On the basis of intuition, there may be people who answer this question either way. Those who opt for "yes" may be hard pressed to explain clearly the basis for their belief. However, we can envisage a phantom individual, call him Proto-Sam, who rides from Truro to Halifax on the first day. Suppose that Proto-Sam is always at exactly the same position that Sam is at exactly twenty-four hours later. So we can regard Proto-Sam as doing a kind of dry run for Sam's return journey. Then it is clear that Sam and Proto-Sam will pass sometime during the first day, and so the answer to the question is indeed "yes".

A more sophisticated version of the same idea arises in the following problem.

5. **The ink blot.**

Some ink is spilled onto a page. It may form a single blob, or may consist of several blobs, but in any case the total area of all the ink spilled is more than 1 square centimeter. Suppose that we have an acetate *page on which is printed a grid of 1 centimeter squares (it consists of ruled horizontal and vertical lines that are 1 centimeter apart). Is it possible to place this grid over the ink blot in such a way that two grid points land on the ink? (A grid point is a point where a horizontal and a vertical line intersect.)*

Certainly the answer is *sometimes* "yes". For example, if the inkblot covers an entire square of a grid and then some. Then it easy to see that we can shift the position of the grid a little so that a grid point rests on part of the blot outside the square and another grid point rests on part of the blot inside the square. Getting a handle on a splattery ink blot is not so easy. One way to look at the situation is impose a grid on the inkblot in any way that we wish, so that the ink adheres to the grid. Now let us slice the grid into unit squares along the grid lines, and translate all the unit squares so that they rest on the same position. Let us call this position the underlying square. Some of this underlying square will rest under parts of the inkblot. The total area of the parts of the inkblot is greater than 1 square unit, the area of the underlying square, so parts of the inkblot on the moved squares must overlap. Stick a pin through the sliced unit squares so that it passes through part of the blot on at least two of them. Now move the sliced squares back to their original positions; the pinholes will appear at the same position of each square, *i.e.* at the grid-points of another grid that we can now place on top in a suitable position.

6. **Spawning squares.**

Write down all the positive whole numbers in order: 1, 2, 3, 4, 5, $\cdots$. Delete every second one, to get just the odd numbers remaining: 1, 3, 5, 7, 9, $\cdots$. Now make a running total of these:

$$1; \quad 4 = 1+3; \quad 9 = 1+3+5; \quad 16 = 1+3+5+7; \quad 25 = 1+3+5+7+9; \quad \cdots$$

What do you notice? What do you think happens in general? Why does it happen?

Once again, write down all the positive whole numbers in order, but this time, we put them in pens (cages) - one in the first pen, two in the second pen, three in the third pen, and so on:

$$1, |\ 2, 3, |\ 4, 5, 6, |\ 7, 8, 9, 10, |\ 11, 12, 13, 14, 15, |\ 16, 17, 18, 19, 20, 21, | \cdots$$

Now we delete every second pen, and get:

$$1, |\ 4, 5, 6, |\ 11, 12, 13, 14, 15, |\ 22, 23, 24, 25, 26, 27, 28, | \cdots$$

Add up the numbers in each pen:

$$1|\ 15|\ 65|\ 175|\ \cdots$$

Finally, make a running total of these sums:

$$1; \quad 16 = 1+15; \quad 81 = 1+15+65; \quad 256 = 1+15+65+175; \ \cdots$$

What do you notice? What do you think happens in general? Why?

This can be generalized. Suppose, in penning off the numbers, you take two more numbers in each pen than you had in the previous pen:

$$1, |\ 2, 3, 4, |\ 5, 6, 7, 8, 9, |\ 10, 11, 12, 13, 14, 15, 16, |$$

$$17, 18, 19, 20, 21, 22, 23, 24, 25, |26, \cdots, 36, | \cdots$$

Experiment and see what you can find out.

These are questions that a student with some mathematical background can do using technical means, algebraic formulae and a proof by induction, once the general statement describing this has been established.

However, there is a way of seeing what is going on that avoids the technical barrier and uses a few ideas in depth. It is a little like a musical composition that elaborates on a simple theme.

The first idea I want to put on the table is that of adding a set of consecutive integers using a process of give and take. That is, we will take a certain amount from one summand and give to another, so that the overall sum remains the same. For example, to add the first seven odd numbers, we note that the middle one is 7 and this is flanked by 5 and 9, which in turn is flanked by 3 and 11, and finally by 1 and 13. The sum does not change is we take 2 from 9 and add it to the 5: $5 + 9 = 7 + 7$. We can do the same all along the sum to find that:

$$1+3+5+7+9+11+13 = 7+7+7+7+7+7+7 = 7 \times 7 = 7^2 \; .$$

Applying this to any number of summands, we can see that the sum of the first n odd numbers is equal to n^2. This is the second idea I want to introduce into the proceedings.

In the second round, we wish to determine the sums:

$$1 = 1$$

$$1 + 15 = 1 + (4+5+6) = 1 + (3+5+7)$$

$$\begin{aligned} 1 + 15 + 65 &= 1 + (4+5+6) + (11+12+13+14+15) \\ &= 1 + (3+5+7) + (9+11+13+15+17) \end{aligned}$$

and so on, where have used the give-and-take principle to rewrite the sums in parenthesis. Now let us interpret these equations: In the first equation, we are adding the first odd number to get 1^2. In the second equation, we are adding the first $1 + 3 = 2^2$ odd numbers to get $(2^2)^2 = 2^4$. In the third equation, we are adding the first $1 + 3 + 5 = 3^2$ odd numbers to get $(3^2)^2 = 3^4$. Now we can infer what is going to happen in general.

At the next stage, we have to take the partial sums (to the end of the parentheses) of the following series:

$$\begin{aligned} &1 + (5+6+7+8+9) + (17+18+19+20+21+22+23+24+25) + \cdots \\ &\quad = 1 + (3+5+7+9+11) + (13+15+17+19+21+23+25+27+29) + \cdots \\ &\quad = 1 + ([3+5] + [7+9+11]) + ([13+15+17+19] + [21+23+25+27+29]) + \cdots \end{aligned}$$

whose partial sums are 1, $(1 + 2 + 3)^2$, $(1 + 2 + 3 + 4 + 5)^2$ and so on.

This case has a bonus. If we take the even rows rather than the odd ones, we find that we want to partial sums of the following series:

$$\begin{aligned} &(2+3+4) + (10+11+12+13+14+15+16) + \cdots \\ &\quad = (1 + [3+5]) + ([7+9+11] + [13+15+17+19]) + \cdots \end{aligned}$$

whose partial sums are $(1 + 2)^2$. $(1 + 2 + 3 + 4)^2$, and so on.

This viewpoint gives us a clearer sense of what is happening than if we had to use a technical proof, where mathematical formalism, while delivering a correct argument, may prevent a complete understanding.

7. **Counting the zeros.**

If you write down all the numbers in order from 1 up to 100, inclusive, you will see that you require 192 digits - 9 digits for the numbers from 1 to 9, inclusive, 180 digits for the 90 two-digit numbers from 10 up to 99, inclusive, and finally three digits for the number 100.

Interestingly enough, if you ask how many zeros you need to write down all the numbers from 1 to 1000, inclusive, you will see that 192 zeros will be needed. No zero is required for any of the single-digit numbers.

However, we will need 9 zeros for two-digit numbers, one for each of the numbers 20, 30, 40, 50, 60, 70, 80, 90. Eighteen more zeros will be needed for the numbers 100, 200, 300, 400, 500, 600, 700, 800, 900. As for the three-digit numbers with a single zero, 81 will have a zero as the middle digit and 81 will have a zero as the last digit. Finally, we need three zeros for 1000.

Make a table to check out the number of digits, and the number of these digits that are zero, when you write out all the numbers from 1 to 10, from 1 to 100, from 1 to 1000, from 1 to 10000. What do you notice? Can you account for it?

Let us get some data down.

Range	Number of digits	Number of zeros
$1 - 10$	11	1
$1 - 100$	192	11
$1 - 1000$	2893	192
$1 - 1000$	38894	2893

We can conjecture on the basis of the evidence so far that the number of digits needed in writing the numbers from 1 to 10^n is equal to the number of zeros used in writing the numbers from 1 to 10^{n+1} for each positive integer n. However, the numbers in the table look decidedly unpromising from the point of view of forming an easy-to-see pattern. However, let us think around the situation.

Look at the numbers from 1 to 100. The three digits for 100 can be matched with the three zeros for 1000. So it is enough to compare the number of digits going from 1 to 99 with the number of zeros going from 1 to 999. Once we realize this, it becomes more natural to compare the number of digits in the single-digit numbers with the number of zeros in the double-digit numbers, and the number of digits in the double-digit numbers with the number of zeros in the triple-digit numbers.

There are nine digits for the single-digit numbers, and clearly nine zeros occur among the double-digit numbers. This is too easy to provide much insight. However, there are ninety two-digit numbers, accounting for a total of 90×2 digits. Now look at the three-digit numbers. This digit 0 never occurs as the left-most digit, so we need only consider the last two digits of these numbers. The insight we need is that, *among the last two digits of all three-digits numbers, each digit occurs equally often.* As a result, the number of zeros is equal to one-tenth of the number of digits beyond the first among the three-digit numbers; this number is equal to $(900 \times 2) \div 10 = 90 \times 2$.

To secure the argument, let us look at numbers with n digits and with $n+1$ digits. The $n-$digit numbers require a total of $(9 \times 10^{n-1}) \times n$ digits. As for the $(n+1)-$digit numbers, there are 9×10^n of them. Of the last n digits (those beyond the first non-zero digit), exactly one-tenth must be zero, so there must be $[n \times (9 \times 10^n)] \div 10 = (9 \times 10^{n-1}) \times n$ zeros in all.

8. **The bodum problem.**

In the diagram below, a square shares a side with an equilateral triangle; all sides have unit length. There is a circle that goes through the apex of the triangle and the bottom two vertices of the square (i.e., the three vertices that are not common to the triangle and the square). What is the radius of this circle?

This is a problem that many students who have studied some high school geometry will be able to get into, but one in which a little insight into the structure will give an immediate solution. Imagine that the triangle atop the square is slid down the sides of the square until its base coincides with the bottom side of the square. It will slide down a distance of one unit. In the final position, its apex will be unit distance from each of the bottom vertices of the square. Thus, the new position of the apex is unit distance from each of the three points described in the problem, so that the three points lie on a circle of radius 1 whose centre is the new position of the apex.

9. **Four points and two lengths.**

Suppose that you have four distinct points on a page. There are six ways of choosing two of them, and each of these six choices of a pair determines a distance between the two points.

If the four points were at the vertices of a square, then there would be two distinct distances determined. Each of the four pairs of adjacent vertices determine one distance and each of the diagonally opposite pairs determines another. How many other configurations of four distinct points can you find for which the six pairs of points determine only two distinct distances?

When this is given to an audience with no example provided, there is a lot of trial and error, and most groups very quickly find the example in which the four points are at the vertices of a square. They will then to go on to find some more examples, but there is one that is very difficult to find. This characteristic makes it a good problem for a mixed group, because some success is easy (and helps define the problem for those who do not quite understand it) while it is a challenge to get the complete story. However, mathematicians are not happy with mere trial and error, and look for some organizing principle that will allow them to go systematically through the possibilities and assure them when they have a complete collection of examples.

There are various ways of getting into the problem. Here is one. Since there are six pairs and only two distances involved, one distance must occur with at least three of the pairs.

Consider first the possibility that the three pairs involve only three points. Then three vertices of the four are vertices of an equilateral triangle. The distance of the fourth vertex from these three is one of two values, so the fourth vertex must be the same distance from two of the first three. Let us give these vertices names. Suppose that A, B, C are the three vertices of an equilateral triangle and that D is the same distance from B and C.

D is the same distance from B and C if and only if it lies on the right bisector of BC, that is, the line which is perpendicular to the segment BC and passes through the midpoint of the segment BC. That each point on the line is equal distance from B and C is seen by regarding the right bisector as an axis of reflection (a mirror) which switches B and C, keeps every point on the bisector fixed and preserves distances between points. Now take a point P that is not on the bisector; suppose that it lies on the same side of the bisector as B. Join P and C and let it intersect the bisector at Q. Then the length of

$$|PC| = |PQ| + |QC| = |PQ| + |QB| > |PB|$$

so that P is strictly closer to B than to C. The bars denote the length of the segment, and the last inequality results from the fact that in the triangle PQB, the length of any side is less than the sum of the lengths of the other two sides.

So where on the right bisector can D be? We could have $|DB| = |DC| = |AB| = |AC| = |BC|$, in which case D is the reflected image of A about the axis BC and we get a rhombus:

On the other hand, the distance between D and B or C could differ from the sidelengths of the equilateral triangle ABC. If it is the same distance from all three vertices of this triangle, then it must be the centroid of the triangle:

Finally, if $|DB| = |DC|$ is different from both $|AB|$ and $|DA|$, then it must happen that $|DA| = |AB| = |BC| = |AC|$. There are two possibilities for this:

Next, consider the possibility that there is no equilateral triangle, and that the three pairs involve all four points. There are, on the face of it, two ways in which this can occur. First DA, DB and DC could have the same length. However, if any of AB, AC, BC shares this length, then we would get an equilateral triangle, while if AB, BC, CA all have the same length differing from $|DA| = |DB| = |DC|$, we again get an equilateral triangle. So this case has already been covered.

Alternatively, we might have $|AB| = |BC| = |CD|$. Since we are forbidding equilateral triangles, we have $|AC| = |BD| \neq |AB|$. There is a final length to consider: $|AD|$. If $|AC| = |AB| = |BC| = |CD|$, we get a square (since the diagonals have to be equal). On the other hand, if $|AD| = |AC| = |BD|$, then we get a trapezoid:

The trapezoid is the case that most groups miss. However, a student in one class found this example right at the start. He considered a regular pentagon, then deleted one of its vertices and the edges and diagonals emanating from it. This is a bit of insight that was new to me!

10. **The seven Königsberg bridges.**

In the eighteenth century, there was a famous problem that had to do with the bridges in the city of Königsberg, located on the shores of the Baltic Sea. The river Pregel wound around an island (marked A in the diagram), and there were seven bridges that connected the various pieces of land (A, B, C, D) *separated by the river. Apparently, it was a pastime of the townspeople to see whether they could plan a walk that would take them over each bridge exactly once, with no bridges missed nor crossed a second time. You are allowed to start and end on any piece of land that you wish. Can the task be done?*

This problem became famous because it came to the attention of the great mathematician, Leonhard Euler (1707-1783), who wrote a paper analyzing it. His research is often regarded as the beginning of the modern field of *graph theory.*

The key is to look at how often each of the regions A, B, C, D have to be visited. Each visit involves at most two bridges, one to enter it and one to leave it (unless you start or finish at that region, in which case only one bridge is crossed). Thus, A must be visited three times, and B, C, D twice each on a complete circuit. Thus, it we list the regions in the order in which we visit them, out list must have *nine* entries. However, there are seven bridges, and each is visited exactly once, so a list of regions visited must have only *eight* entries.

Thus, we get two contradictory statements. Our assumption that the problem can be solved is false.

This problems illustrates an important type of result occurring in mathematics as well as a standard proof technique, *proof by contradiction.* Very often, a mathematical theorem asserts that something cannot be done. In particular, this occurs in decision theory where an apparently reasonable set of criteria for a political situation cannot be all realized at once.

11. **Two pairs of numbers**

Find two pairs of positive whole numbers for which the sum of each of the pairs is equal to the product of the others.

The key to this problem is to note that one generally expects the product of two positive whole numbers to exceed the sum. This is not true when one of the numbers is 1, for then the sum is one more than the product. But suppose that both the numbers are greater than 1. Then the sum of the two numbers does not exceed twice the larger, which in turn does not exceed the product.

This verbal argument might require a moment or two of thought in order to comprehend it, and we can see how a little algebraic notation will allow it to be grasped more easily. Let the two numbers be x and y, with $2 \leq x \leq y$. Then

$$x + y \leq 2y \leq xy$$

and equality occurs exactly when $x = y = 2$. From this we see that two pairs that satisfy what we want are $\{(2, 2), (2, 2)\}$. What are other possibilities?

For the other possibilities, it must be that for one of the pairs, the product does not exceed the sum, while for the other, the sum does not exceed the product. Consider the pair for which the product does not exceed the sum. If we put aside the pair $(2, 2)$, we see that one number in the pair must be equal to 1, so that the sum is one more than the product. Thus, for the other pair, the product is one more than the sum. What must the second pair be?

If we go back to the inequalily

$$x + y \leq 2y \leq xy$$

and ask that the difference between xy and $x + y$ be only 1, then we can see that the only possibility is that $(x, y) = (2, 3)$. ¿From this, we can discover that the only other possibility of the two pairs is $\{(2, 3), (1, 5)\}$.

We can use algebra to give a more sophisticated solution of the problem. Let the two pairs be $\{(x, y), (u, v)\}$. Then the conditions of the problem can be stated:

$$x + y = uv \ ;$$

$$xy = u + v \ .$$

Subtracting one equation from the other and arranging terms leads to

$$xy - (x + y) + uv - (u + v) = 0$$

whereupon

$$(x - 1)(y - 1) + (u - 1)(v - 1) = 2 \ .$$

Since x, y, u, v are all positive whole numbers, both terms on the left side are nonnegative integers. Thus the equation expresses the fact that the sum of two nonnegative integers is equal to 2. There are only two ways in which this can happen: $2 = 0 + 2 = 1 + 1$. In the first case, we have, say, $x = 0, u = 2, v = 3$ and can go back to the original conditions to find that $y = 5$. In the second case, we must have $x = y = u = v = 2$. Thus, we get the two possibilities we found before.

12. **The sliding ladder.**

A ladder is flush against a vertical wall and it rests on a horizontal floor, at right angles to the wall. Suddenly, the bottom of the ladder slides outwards along the floor from the wall, and the top of the ladder slides down the wall. What is the path traced out by the point exactly halfway up the ladder?

Depending on your viewpoint, one can presume the path to be either of the two possibilities:

In favour of the first is the perception that as the ladder slides from the wall, the base moves horizontally while the top moves hardly at all, so that the middle of the ladder begins by moving parallel to the floor. However, the second path also seems reasonable since the ladder seems to move in a scooping way.

As we shall see, the first idea is correct. Probably, those expecting the second path are psychologically confusing the path of the midpoint with a curve called the *envelope* of the positions of the ladder. This is a curve for which all the segments representing the positions of the ladder are tangents. (If the ladder is of unit length and the wall and ground are the coordinate axes, this curve has equation $x^{2/3} + y^{2/3} = 1$.)

There is an insightful way of seeing what the path of the midpoint is. The wall, ground and ladder at each instant form a right triangle. This consitutes half of a rectangle for which the ladder is a diagonal. The two diagonals of the rectangle are of equal length and bisect each other, so that the distance from the wall-ground intersection to the middle of theladder is equal to half the length of the ladder.

Thus, the midpoint of the ladder is a constant distance from the wall-ground intersection and so moves on a circle.

Interestingly enough, the path is the same had the ladder tipped over to the ground with one end fixed at the wall-ground intersection. This is illustrated very nicely by a standard fold-up ironing board with crossed legs.

13. **Quadrilaterals.**

I drew a convex quadrilateral. (This is a four-sided figure that has the property that the segment joining any two vertices is either a side or a diagonal that lies entirely inside the figure.) *Then I joined the midpoints of adjacent sides.*

This seemed to give me a parallelogram. I wondered whether a parallelogram would emerge if I started with a different convex quadrilateral. How much of the area of the quadrilateral does the central figure that looks like a parallelogram take up? Look at some examples of your own and see what you think.

The internal figure is indeed a parallelogram, whose sides are parallel to the diagonals of the given quadrilateral. To see this, consider one of the triangles ABC bounded by two adjacent sides AB, AC and a diagonal BC of the quadrilateral. One side of the inner figure joins the midpoints P, Q of the sides of this triangle.

We can imagine a scale expansion that fixed A and increases distances from A be a factor of 2. This expansion takes P to B, Q to C and the segment PQ to the parallel segment BC. Triangle APQ is a scaled-down version of triangle ABC.

The area of the inner parallelogram is equal to half the area of the quadrilateral. This can be demonstrated using a cardboard model. The triangular "ears" of the quadrilateral external to the parallelogram can be arranged to cover the parallelogram. Triangles 1 and 2 can be translated into new positions while triangles 3 and 4 can be rotated. (A little analysis is needed to verify that the four triangles in their final positions neither overlap not leave part of the parallelogram uncovered.)

14. **Ten numbers**

I have a set of double-nine dominoes. I picked out ten at random and recorded a set of two-digit numbers, based on the two sides of the double-face:

46, 91, 49, 36, 27, 32, 71, 61, 55, 42.

I noticed that I could find two non-overlapping sets of them that had the same sum, namely $\{91\}$ *and* $\{36, 55\}$.

I performed the experiment again, and this time I got:

11, 88, 40, 29, 22, 2, 26, 27, 31, 70.

Was it possible to find two non-overlapping sets of these with the same sum? It was a little harder to find them, but here they are: $\{40, 22\}$ *and* $\{29, 31, 2\}$.

Now, I wondered, suppose I took any ten numbers less than 100. Would it always be possible to find two non-overlapping sets from them that had the same sum? Why don't you check out some sets of your own? (One way to get samples is to take the last two digits of phone numbers in a list or numbers from the business page of the newspaper.)

Another possibility is for one person to make up a set of ten distinct numbers less than 100 and challenge *a second to find two non-overlapping sets with the same sum. The first person wins if the second fails to find such sets; otherwise, the second person wins.*

One can always find two sets with the same sum. Behind the argument for this is a very famous and simple principle enunciated early in the nineteenth century by the mathematician, Gustav Lejeune Dirichlet (1805-1859) (who was a brother-in-law of Felix Mendelssohn). The Pigeonhole Principle states that, if you distibute letters into pigeonholes and the number of letters exceeds the number of pigeonholes, then some pigeonhole must contain more than one letter. Many mathematicians (including Dirichlet) have used this to obtain deep and significant results, but it can be used to give surprising assertions in less considerable situations. This situation is an example.

To each of the subsets of the ten numbers, we can attach its sum. Since each number is less than 100, the sum must be less than 1000. Imagine that we have one thousand pigeonholes labelled from 1 to 1000, and we assign each set to the pigeonhole according to its sum.

How many possible subsets of the ten numbers are there? We can make up a subset by looking at each number among the ten in turn and deciding whether or not to include it. This is a choice among two options made ten times, so together we have $2^{10} = 1024$ possibilities. (To help see this, look at the number of subsets of three elements $\{a, b, c\}$; the subsets are $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$ and the empty set with no elements.) For the ten numbers, this method of selection includes the empty set where every number is refused. So, in all there are 1023 subsets to be assigned to our 1000 pigeonholes.

Thus by the Pigeonhole Principle, two sets are assigned to the same pigeonhole; these two sets have the same sum. But we are not quite finished! The two sets may have elements in common. But this is all right. Simply remove the common elements from both sets to obtain two non-overlapping sets with the same sum.

15. **Matchmaking.**

Take a group with an even number of people and divide it into two parts; let us call the subsets A and B. Each person in A makes a list in some order of the persons in B, and, likewise, each person in B makes a list in order of the persons in A. Your job is to pair off the two sets A and B in such a way that each person in A is matched to a distinct person in B so that this condition is satisfied:

There should be no person in A and no person in B who both are higher on the other's list that the people that they have been assigned to.

[One way to formulate this situation is as a *marriage problem.* The two sets consist of equal numbers of men and women, and each person lists the persons of the opposite sex in order of preference. The job of the matchmaker is to ensure that there are no possibilities of divorce. That is, you cannot find an unmatched pair, each of whom prefers the other to the spouses that they have been assigned.]

For example, Suppose $A = \{p, q, r\}$ and $B = \{u, v, w\}$ and the preference lists are as follows:
$(p : u, v, w)$; $(q : u, w, v)$; $(r : v, u, w)$
$(u : p, q, r)$; $(v : p, r, q)$; $(w : r, q, p)$
One possible pairing is (p, v), (q, u), (r, w). However, this is unsatisfactory, since p and u rank each other more highly than their own assigned matches. However, (p, u), (q, w), (r, v) is acceptable. Does (p, u), (q, v), (r, w) also work?

Experiment with other possibilities to see whether a suitable pairing can always be made.

A matching of the required type is, in fact, always possible. There are many ways to proceed; here is one. The process requires several rounds. In the first round, each person in A goes to ("proposes") to that person from B at the top of her list. B may receive several "offers"; if so, B accepts the offer from that proposer from A highest on his list, and rejects all others. If at the end of the first round, each person in A has been accepted, the process terminates. Otherwise, some people in A have been refused; each such person crosses from her list the person in B who turned her down, and we proceed to the second round.

In the second round, each person in A who has not yet been accepted by a person in B goes to the top person in B not crossed off on her list. Each person in B accepts the highest person from A (including, if applicable, the one accepted in the earlier round) on his list. If there are no rejectees, the process terminates. Otherwise, we move to another round, with rejectees crossing their rejectors from their lists. Each subsequent round proceeds in the same way.

How can we be sure that the process terminates? A new round is necessaary because there has been a rejection; each rejection results in a person being crossed from someone's list. There are only finitely many crossings that can occur and so only finitely many rejections and finitely many rounds.

We show that this matching has the desired property. Suppose that p is paired with u and q is paired with v (p and q are in A; u and v are in B). However, it may be that p prefers v to u. If this is the case, p will have approached v before approaching u, since v is higher in p's list. But v must rejected p in favour of someone more preferred, so that in the end v prefers q to p. Thus, v has no incentive to defect to p.

On the other hand, suppose that u prefers q to p. If u had been approached by q, then u would either have accepted q or given up q for someone better (not p). So u could not have been approached by q. But then q would have approached and been accepted by someone better (eventually v).

This problem and its solution is pure mathematics, even though it does not involve any manipulation with numbers or equations. It focusses on an essential characteristic of mathematics as a process of analysis and argumentation. Because of this, there are no technical requirements and the result is accessible to anyone who is capable of logical reasoning.

I hope that I have been able to achieve two goals with this collection. The first is to make the point that there is interesting mathematics that can be explored by those who do not have a particular background in the subject. The second is that problems can be given to lay people that reflect the essential characteristics of the subject, and move them beyond the perception that mathematics is simply computation and the application of formulae. It involves understanding, judgment and creativity.