MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Ascension of Our Lord Secondary School, Mississauga), Eric Robert (Leo Hayes High School, Fredericton), Larry Rice (University of Waterloo), and Ron Lancaster (University of Toronto).

Mayhem Problems

Please send your solutions to the problems in this edition by **1 May 2008**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M326. *Proposed by the Mayhem Staff.*

The notation $\underline{a}89\underline{b}$ means the four-digit (base 10) integer whose thousands digit is a, whose hundreds digit is 8, whose tens digit is 9, and whose units digit is b. Determine all pairs of non-zero digits a and b such that $\underline{a}89\underline{b} - 5904 = \underline{b}98\underline{a}$.

M327. *Proposed by Lino Demasi, student, University of Waterloo, Waterloo, ON.*

Kaitlyn bought a new eraser. Her new eraser is in the shape of a rectangular prism. She calculates the lengths of the diagonals of the faces to be 10, $\sqrt{61}$, and $\sqrt{89}$. What is the volume of Kaitlyn's eraser?

M328. *Proposed by Hugo Cuéllar, Columbia Aprendiendo, Zipaquirá, Colombia.*

Prove that, if from any positive integer we subtract the sum of each of its digits raised to any odd power (not necessarily the same), then the result is always a multiple of 3.

M329. *Proposed by the Mayhem Staff.*

Determine the value of

$$\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 89^\circ + \cos^2 90^\circ .$$

M330. *Proposed by the Mayhem Staff.*

If n is a positive integer, the n^{th} triangular number is defined as $T_n = 1 + 2 + \cdots + (n-1) + n = \frac{1}{2}n(n+1)$. Determine all pairs of triangular numbers whose difference is 2008.

M331. *Proposed by the Mayhem Staff.*

In trapezoid $ABCD$, side AB is parallel to DC, and diagonals AC and BD intersect at P.

(a) If the area of $\triangle APB$ is 4 and the area of $\triangle DPC$ is 9,

  (i) prove that $AP : PC = 2 : 3$,
  (ii) explain why the ratio of the area of $\triangle BPC$ to the area of $\triangle BPA$ equals $3 : 2$, and
  (iii) determine the area of trapezoid $ABCD$.

(b) If the area of $\triangle APB$ is x and the area of $\triangle DPC$ is y, determine the area of trapezoid $ABCD$ in terms of x and y.

M326. *Proposé par l'Équipe de Mayhem.*

L'expression $\underline{a}89\underline{b}$ dénote un entier de quatre chiffres (écrit en base 10) dont le chiffre des milliers est a, celui des centaines est 8, celui des dizaines est 9, et celui des unités est b. Déterminer toutes les paires de chiffres non nuls a et b tels que $\underline{a}89\underline{b} - 5904 = \underline{b}98\underline{a}$.

M327. *Proposé par Lino Demasi, étudiant, Université de Waterloo, Waterloo, ON.*

Catherine s'est acheté une nouvelle gomme à effacer, en forme de prisme rectangulaire, dont les diagonales des faces mesurent 10, $\sqrt{61}$, et $\sqrt{89}$. Quel est le volume de la gomme de Catherine ?

M328. *Proposé par Hugo Cuéllar, Columbia Aprendiendo, Zipaquirá, Colombie.*

Montrer que si l'on soustrait d'un entier positif arbitraire, la somme de chacun de ses chiffres élevé à une puissance impaire quelconque (pas nécessairement la même),on obtient toujours un multiple de 3.

M329. *Proposé par l'Équipe de Mayhem.*

Déterminer la valeur de

$$\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 89^\circ + \cos^2 90^\circ .$$

M330. *Proposé par l'Équipe de Mayhem.*

Soit n un entier positif. On définit le $n^{\text{ième}}$ nombre triangulaire T_n comme la somme $T_n = 1 + 2 + \cdots + (n-1) + n = \frac{1}{2}n(n+1)$. Déterminer toutes les paires de nombres triangulaires dont la différence est 2008.

M331. *Proposé par l'Équipe de Mayhem.*

Soit un trapèze $ABCD$ dont les côtés AB et DC sont parallèles et les diagonales AC et BD se coupent en P.

(a) Si l'aire du triangle APB est 4 et celle du triangle DPC est 9,

(i) montrer que $AP : PC = 2 : 3$,

(ii) expliquer pourquoi le rapport des surfaces des triangles BPC et BPA est égal à $3 : 2$, et

(iii) déterminer l'aire du trapèze $ABCD$.

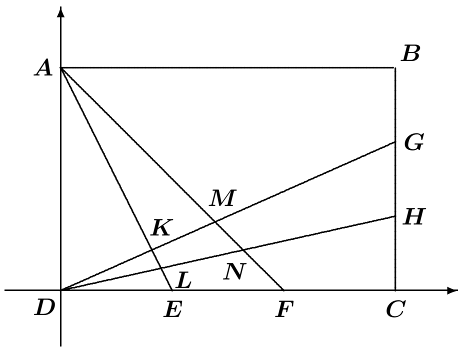
(b) Si l'aire du triangle APB est x et celle du triangle DPC est y, déterminer l'aire du trapèze $ABCD$ en fonction de x et y.

Mayhem Solutions

M276. *Proposed by Babis Stergiou, Chalkida, Greece.*

In rectangle $ABCD$, points E and F divide side DC into three equal parts $DE = EF = FC$, and points G and H divide side BC into three equal parts $BG = GH = HC$. The line AE cuts the lines DG and DH at points K and L, respectively. Similarly, the line AF cuts the lines DG and DH at points M and N, respectively. Show that $KN \parallel CD$.

Combined solution from George Apostolopoulos, Mesologi, Greece; Taichi Maekawa, Takatsuki City, Osaka, Japan; Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Justin Yang, student, Lord Byng Secondary School, Vancouver, BC.



Rectangle $ABCD$ is placed into the Cartesian plane such that its vertices are as follows: $A(0, 3b)$, $B(3a, 3b)$, $C(3a, 0)$, and $D(0, 0)$. The coordinates of the remaining points are then $G(3a, 2b)$, $H(3a, b)$, $F(2a, 0)$, and $E(a, 0)$. The equations of lines DH and AF are obtained as $y = \frac{b}{3a}\,x$ and $y = -\frac{3b}{2a}\,x + 3b$, respectively. Solving these, we get the y-coordinate of their point of intersection, N, as $y = \frac{6b}{11}$. Similarly, the equations of lines DG and AE are found to be $y = \frac{2b}{3a}\,x$ and $y = -\frac{3b}{a}\,x + 3b$, respectively. This yields the y-coordinate of their point of intersection, K, as $y = \frac{6b}{11}$.

Hence, since points K and N are equidistant from side DC, line segments KN and DC must be parallel.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; and TITU ZVONARU, Comănești, Romania.

M277. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Let $f(n,k)$ be the number of ways of distributing k candies to n children so that each child receives at most two candies. For example, if $n=3$, then $f(3,7)=0$, $f(3,6)=1$, and $f(3,4)=6$. Determine the value of

$$f(2007,1)+f(2007,4)+f(2007,7)+\cdots+f(2007,4012)\,.$$

Solution by Daniel Tsai, student, Taipei American School, Taipei, Taiwan.

First note that $f(n,1)=n$ and $f(n,2)=n+C_2^n$ for integers $n\geq 1$. Define $f(n,k)=0$ for integers $n\geq 1$ and $k<0$. Then we can arrange the $f(n,k)$s as:

$$\begin{array}{ccccccc} \cdots & f(1,-1) & f(1,0) & f(1,1) & f(1,2) & f(1,3) & \cdots \\ \cdots & f(2,0) & f(2,1) & f(2,2) & f(2,3) & f(2,4) & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

or equivalently as:

$$\begin{array}{ccccccc} \cdots & 0 & 1 & 1 & 1 & 0 & \cdots \\ \cdots & 1 & 2 & 3 & 2 & 1 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

The number of ways of distributing k candies to n children is the same as the number of ways of distributing 0 candies to a certain child while distributing k candies to the $n-1$ others, plus the number of ways of distributing 1 candy to the same child while distributing $k-1$ candies to the others, plus the number of ways of distributing 2 candies to the same child while distributing $k-2$ candies to the others; that is,

$$f(n,k)\;=\;f(n-1,k)+f(n-1,k-1)+f(n-1,k-2)\,.$$

Let R_n be the sum of the numbers in the n^{th} row of the above table for integers $n\geq 1$. Then $R_{n+1}=3R_n$ for integers $n\geq 1$, and since $R_1=3$, we get $R_n=3^n$ for integers $n\geq 1$. Finally, we find that $f(2006,k)=0$ for integers $k<0$ or $k>4012$. Therefore,

$$\begin{aligned} & f(2007,1)+f(2007,4)+f(2007,7)+\cdots+f(2007,4012) \\ & \qquad =\;\sum_{k=0}^{4012} f(2006,k)\;=\;R_{2006}\;=\;3^{2006}\,. \end{aligned}$$

Also solved by DIVYANSHU RANJAN, high school student, Delhi, India; and JUSTIN YANG, student, Lord Byng Secondary School, Vancouver, BC. One incomplete solution was also submitted.

M278. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Find sixteen 16–digit palindromes, in each of which the product of the non-zero digits and the sum of the digits are both equal to 16. How many such numbers are there?

A composite of solutions from Daniel Tsai, student, Taipei American School, Taipei, Taiwan; Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA; and Justin Yang, student, Lord Byng Secondary School, Vancouver, BC.

A 16–digit palindrome must have the form $d_1d_2\ldots d_8d_8\cdots d_2d_1$, with $d_i \in \{0, 1, 2, \ldots, 9\}$ for $1 \le i \le 8$, and $d_1 \neq 0$. For the sum of its digits to be 16, the sum $d_1 + d_2 + \cdots + d_8$ must be 8. Simultaneously, for the product of the non-zero digits of the palindrome to be 16, the product of the non-zero digits among its first eight digits must be 4. It can easily be seen that the first eight digits of the required palindromes must be arrangements, either of digits 2, 2, 1, 1, 1, 1, 0, 0 or of digits 4, 1, 1, 1, 1, 0, 0, 0, with 0 excluded as first digit. The following are 16 such palindromes:

$$\begin{array}{ll}
4111100000011114\,, & 1411100000011141\,,\\
1141100000011411\,, & 1114001001004111\,,\\
1100141001410011\,, & 1010410110140101\,,\\
1100140110410011\,, & 1004111001114001\,,\\
1041110000111401\,, & 4000111111110004\,,\\
2211110000111122\,, & 2121110000111212\,,\\
2112110000112112\,, & 2111210000121112\,,\\
2111120000211112\,, & 2011121001211102\,.
\end{array}$$

The total number of such palindromes, with 0 excluded as first digit, is obtained as follows:

If the first 8 digits of the palindrome contain two 2s, four 1s, and two 0s, then the number of arrangements of these 8 digits is

$$\frac{6(7!)}{2!2!4!} = 315\,.$$

If the first 8 digits of the palindrome contain one 4, four 1s, and three 0s, then the number of arrangements is

$$\frac{5(7!)}{4!3!} = 175\,.$$

Thus, there are $315 + 175 = 490$ such numbers.

Also solved by ELIAS C. BUISSANT DES AMORIE, CJ Castricum, the Netherlands; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA. There were also two incorrect solutions submitted.

M279. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine an infinite set of rational number solutions (α, β) to the equation $\alpha^2 + \beta^2 = \alpha^3 + \beta^3$.

Composite solution from Hasan Denker, Istanbul, Turkey; and Geoffrey A. Kandall, Hamden, CT, USA.

The given equation is equivalent to

$$\begin{aligned}(\alpha+\beta)^2 - 2\alpha\beta &= \alpha^2+\beta^2 = \alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2) \\ &= (\alpha+\beta)\left[(\alpha+\beta)^2 - 3\alpha\beta\right].\end{aligned}$$

Setting $\alpha + \beta = p$ and $\alpha\beta = q$, we obtain $p^2 - 2q = p(p^2 - 3q)$. Solving this equation for q yields

$$q = \frac{p^2(p-1)}{3p-2}. \qquad (1)$$

Note that α and β are roots of the quadratic equation $x^2 - px + q = 0$. The discriminant of this equation is $D = p^2 - 4q$. Using (1), we obtain

$$D = \frac{p^2(2-p)}{3p-2}.$$

Since we want α and β to be rational numbers, we must have $\sqrt{D}$ rational. Let $D = p^2k^2$, where $k \in \mathbb{Q}$ and $k^2 = \dfrac{2-p}{3p-2}$. Solving this equation for p yields $p = \dfrac{2(1+k^2)}{1+3k^2}$.

Hence, an infinite set of solutions to the equation $\alpha^2 + \beta^2 = \alpha^3 + \beta^3$ is given by the set

$$\begin{aligned}&\left\{(\alpha,\beta) = \left(\tfrac{p \pm pk}{2}, \tfrac{p \mp pk}{2}\right) : k \in \mathbb{Q}\right\} \\ &\qquad = \left\{(\alpha,\beta) = \left(\frac{(1+k^2)(1\pm k)}{1+3k^2}, \frac{(1+k^2)(1\mp k)}{1+3k^2}\right) : k \in \mathbb{Q}\right\}.\end{aligned}$$

Also solved by MIHÁLY BENCZE, Brasov, Romania; ELIAS C. BUISSANT DES AMORIE, CJ Castricum, the Netherlands; SAMUEL GÓMEZ MORENO, Universidad de Jaén, Jaén, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VEDULA N. MURTY, Dover, PA, USA; and DANIEL TSAI, student, Taipei American School, Taipei, Taiwan. There was one incorrect solution submitted.

M280. *Proposed by the Mayhem Staff.*

An equilateral triangle lies in the plane with two of its vertices at points $(0, 0)$ and $(0, n)$. Determine the number of points (x, y) with integer coordinates which lie in the interior of the triangle.

Solution by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.

We will consider the triangle with the third vertex in the first quadrant. If $n = 1$, it is clear that there are no points with integer coordinates in the interior of the triangle; if $n = 2$, it is also clear that there is exactly one such point in the interior of the triangle, namely $(1, 1)$. Let us now assume that $n \geq 3$.

Two of the lines which determine the triangle are $y = \left(\tan\frac{\pi}{6}\right)x = \frac{\sqrt{3}}{3}\,x$ and $y = n - \left(\tan\frac{\pi}{6}\right)x = n - \frac{\sqrt{3}}{3}\,x$; thus, the third vertex has coordinates $\left(\frac{\sqrt{3}}{2}\,n, \frac{1}{2}\,n\right)$. For each positive integer i, the number $\sqrt{3}i$ is not an integer; hence, there are $\lfloor\sqrt{3}i\rfloor$ lattice points inside the triangle and on the line $y = i$. By symmetry with respect to $y = n/2$, we see that, for odd n with $n \geq 3$, the number of lattice points is

$$2\sum_{i=1}^{(n-1)/2}\lfloor\sqrt{3}i\rfloor\,.$$

In the same way, for even n with $n \geq 4$, the number of lattice points is

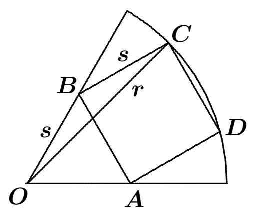
$$\left(2\sum_{i=1}^{(n/2)-1}\lfloor\sqrt{3}i\rfloor\right) + \lfloor\sqrt{3}n/2\rfloor\,.$$

HASAN DENKER, Istanbul, Turkey provided a very good approximation using Pick's Formula. Two incorrect solutions were also submitted.

M281. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A square of side length s is inscribed symmetrically inside a sector of a circle with radius of length r and central angle of 60°, such that two vertices lie on the straight sides of the sector and two vertices lie on the circular arc of the sector. Determine the exact value of s/r.

Solution by Nick Wilson, student, Valley Catholic School, Beaverton, OR, USA.



By symmetry, we see that $OA = OB$. Since $\angle AOB = 60^\circ$, we see that $\triangle AOB$ is equilateral. Thus, $OB = AB = BC = s$. We can now use the Law of Cosines on $\angle OBC$ to successively obtain:

$$\begin{aligned} r^2 &= s^2 + s^2 - 2s^2 \cos 150^\circ = s^2\left(2 + 2 \cdot \frac{\sqrt{3}}{2}\right), \\ \left(\frac{r}{s}\right)^2 &= 2 + \sqrt{3}, \\ \frac{s}{r} &= \frac{1}{\sqrt{2+\sqrt{3}}} = \sqrt{2 - \sqrt{3}}. \end{aligned}$$

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; DANIEL TSAI, student, Taipei American School, Taipei, Taiwan; JUSTIN YANG, student, Lord Byng Secondary School, Vancouver, BC; and TITU ZVONARU, Comănești, Romania. Two incorrect solutions were also submitted.

Problem of the Month

Ian VanderBurgh

After a long winter's break, it's time to stop playing games and, well, . . . start playing games!

Problem (2006 Cayley Contest)
Anne and Brenda play a game which begins with a pile of n toothpicks. They alternate turns with Anne going first. On each player's turn, she must remove 1, 3, or 4 toothpicks from the pile. The player who removes the last toothpick wins the game. For which of the values of n from 31 to 35 inclusive does Brenda have a winning strategy?

With this game problem, as with any such problem, the best thing to do initially is to get out a pencil and a piece of paper (or better yet, a pile of toothpicks) and give it a try, starting with some small values for n.

Even before we do this, though, it's probably worth remembering what it means to have a "winning strategy" in such a game. A player has a winning strategy if, regardless of what the other player does, there are moves that she can make that *guarantee* that she will win. (In fact, it's worth checking out some back issues of *Mayhem* at this point—my trusty colleague, John Grant McLoughlin, wrote a couple of *Pólya's Paragon* columns in 2006 about mathematical games [2006 : 275–276, 369–371].)

Let's try some small values of n to see if we can get a feel for this game. We'll abbreviate the players' names (conveniently) as A and B. I would suggest that you get out a pencil and a piece of paper and try the cases $n = 1$ to $n = 6$ before we go through this together.

If $n = 1$, A wins by immediately taking 1 (leaving 0).

If $n = 2$, A cannot take 3 or 4; thus, A must take 1, leaving 1, and B wins by taking 1 (leaving 0).

If $n = 3$ or $n = 4$, A wins by immediately taking 3 or 4, respectively.

The value $n = 5$ is where things start to get more interesting. If A removes 1, B receives a pile of 4, and wins by taking 4 (leaving 0). So A doesn't want to remove 1. If A removes 4, B receives a pile of 1, and wins by taking 1. So A doesn't want to remove 4 either. If A removes 3, B receives a pile of 2. Here, B can only remove 1 (not 3 or 4), and A then receives a pile of 1, and so wins by removing 1.

What does this case tell us? Who has the winning strategy? If A chooses 1 or 4, she loses, but if she chooses 3, she wins. Thus, A has a winning strategy, as she controls her own fate by choosing first (and hopefully choosing 3).

When $n = 5$, A's winning strategy was to remove 3, leaving B with 2. We showed by looking at what B and A can remove that A must win. But is there a way of looking at this that might be easier to generalize? This could be really useful. Think about this as we're looking at the case of $n = 6$.

If $n = 6$, A can remove 1, 3, or 4, leaving B with 5, 3, or 2, respectively. If A removes 1 leaving B with 5, then B could be clever and remove 3, leaving A with 2. But we saw above that A loses when choosing from a pile of size 2. Thus, if A removes 1, then B can force A to lose. If A removes 3 leaving B with 3, then B can remove 3 and win. That is, if A removes 3, then B can force A to lose. If A removes 4 leaving B with 2, then B will lose as the first person choosing from a pile of size 2 will lose.

Wait! That's the key right there. Starting with $n = 6$, A can reduce the pile to 2, 3, or 5. But we've already looked at these cases. The first player to choose should win starting with a pile of 3 or 5, but should be forced to lose starting with a pile of 2. After A has chosen and passed the pile to B, then B will be the first one to choose (the tables have been turned). So if A chooses 4, then B will lose.

Can we generalize this now? Starting with a pile of size n, if A can choose in such a way as to reduce the pile to one where the first player to choose (now B) does not have a winning strategy, then A will win. If all three of the positions to which A can reduce the pile are positions where the first player to choose (now B) has a winning strategy, then A will lose (as B can follow a winning strategy no matter how A chooses initially).

If $n = 7$, then A can remove 1, 3, or 4, leaving B with 6, 4, or 3, respectively, and all three of these possibilities have a winning strategy for the player who chooses first (here, B). Hence, B can force A to lose, so A does not have a winning strategy for $n = 7$.

We're now ready to write down a solution to the original problem. You will see that our solution will be quite short because of all of the work we've done in advance. (This is a great technique—do the legwork beforehand so that things are simpler later on.) If you don't feel like you've got a grip on the problem, then try doing some larger cases, such as $n = 8$ to $n = 12$. Then read on.

Solution: From above, A has a winning strategy if $n = 1, 3, 4, 5, 6$, but does not have a winning strategy if $n = 2$ or $n = 7$.

Starting with a pile of size n, A must reduce the pile to one of size $n-1$, $n-3$, or $n-4$ and pass it to B. If the first person to choose (now B) has a winning strategy starting with a pile of each of these sizes, then A will lose. In other words, if A has a winning strategy starting with piles of size $n-1$, $n-3$, and $n-4$, then A will lose starting with a pile of size n, since B can implement A's strategy for the smaller pile and win, no matter what A does.

If one or more of these pile sizes are such that the first person does not have a winning strategy, then A should reduce to this size, which prevents B from being able to win. Thus, A herself will win.

From $n = 8$, A can reduce to 7, 5, or 4. Since the first player does not win when starting with 7, then A wins for $n = 8$ by taking 1 toothpick and reducing the pile to 7.

For $n = 9$, A can reduce the pile to 8, 6, or 5. Since the first player has a winning strategy for each of these sizes, we see that A loses when $n = 9$.

For $n = 10$ and $n = 11$, A can reduce the pile to 7 by removing 3 and 4, respectively. Since the first player does not have a winning strategy starting with 7, then A wins starting with $n = 10$ and $n = 11$.

For $n = 12$ and $n = 13$, A can reduce the pile to 9 by removing 3 and 4, respectively. Since the first player does not have a winning strategy starting with 9, then A wins starting with $n = 12$ and $n = 13$.

Continuing to examine cases in this way, we can list the winning and losing starting positions for A:

Winning: 15, 17, 18, 19, 20, 22, 24, 25, 26, 27, 29, 31, 32, 33, 34, ...

Losing: 14, 16, 21, 23, 28, 30, 35, ...

Therefore (to answer the question that was asked!), Anne has a winning strategy when n is 31, 32, 33, and 34, and does not when n is 35.

This is an appealing problem in many ways. There are several interesting things to think about—what a winning strategy means, how winning strategies for some positions correspond to winning strategies at other positions, and so on.

There are also some interesting extensions here for those of you who like to look a little bit beyond. Can you figure out who has a winning strategy if $n = 100$? Can you determine a complete list of winning positions for the two players? What happens if the players can remove 1, 2, or 4 instead of 1, 3, or 4? How about 1, 3, or 6? There is always more to think about!