

Problem of the Month

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After a long winter's break, it's time to stop playing games and, well, . . . start playing games!

Problem (2006 Cayley Contest)

Anne and Brenda play a game which begins with a pile of n toothpicks. They alternate turns with Anne going first. On each player's turn, she must remove 1, 3, or 4 toothpicks from the pile. The player who removes the last toothpick wins the game. For which of the values of n from 31 to 35 inclusive does Brenda have a winning strategy?

With this game problem, as with any such problem, the best thing to do initially is to get out a pencil and a piece of paper (or better yet, a pile of toothpicks) and give it a try, starting with some small values for n .

Even before we do this, though, it's probably worth remembering what it means to have a "winning strategy" in such a game. A player has a winning strategy if, regardless of what the other player does, there are moves that she can make that *guarantee* that she will win. (In fact, it's worth checking out some back issues of *Mayhem* at this point—my trusty colleague, John Grant McLoughlin, wrote a couple of *Pólya's Paragon* columns in 2006 about mathematical games [2006 : 275–276, 369–371].)

Let's try some small values of n to see if we can get a feel for this game. We'll abbreviate the players' names (conveniently) as A and B . I would suggest that you get out a pencil and a piece of paper and try the cases $n = 1$ to $n = 6$ before we go through this together.

If $n = 1$, A wins by immediately taking 1 (leaving 0).

If $n = 2$, A cannot take 3 or 4; thus, A must take 1, leaving 1, and B wins by taking 1 (leaving 0).

If $n = 3$ or $n = 4$, A wins by immediately taking 3 or 4, respectively.

The value $n = 5$ is where things start to get more interesting. If A removes 1, B receives a pile of 4, and wins by taking 4 (leaving 0). So A doesn't want to remove 1. If A removes 4, B receives a pile of 1, and wins by taking 1. So A doesn't want to remove 4 either. If A removes 3, B receives a pile of 2. Here, B can only remove 1 (not 3 or 4), and A then receives a pile of 1, and so wins by removing 1.

What does this case tell us? Who has the winning strategy? If A chooses 1 or 4, she loses, but if she chooses 3, she wins. Thus, A has a winning strategy, as she controls her own fate by choosing first (and hopefully choosing 3).

When $n = 5$, A 's winning strategy was to remove 3, leaving B with 2. We showed by looking at what B and A can remove that A must win. But is there a way of looking at this that might be easier to generalize? This could be really useful. Think about this as we're looking at the case of $n = 6$.

If $n = 6$, A can remove 1, 3, or 4, leaving B with 5, 3, or 2, respectively. If A removes 1 leaving B with 5, then B could be clever and remove 3, leaving A with 2. But we saw above that A loses when choosing from a pile of size 2. Thus, if A removes 1, then B can force A to lose. If A removes 3 leaving B with 3, then B can remove 3 and win. That is, if A removes 3, then B can force A to lose. If A removes 4 leaving B with 2, then B will lose as the first person choosing from a pile of size 2 will lose.

Wait! That's the key right there. Starting with $n = 6$, A can reduce the pile to 2, 3, or 5. But we've already looked at these cases. The first player to choose should win starting with a pile of 3 or 5, but should be forced to lose starting with a pile of 2. After A has chosen and passed the pile to B , then B will be the first one to choose (the tables have been turned). So if A chooses 4, then B will lose.

Can we generalize this now? Starting with a pile of size n , if A can choose in such a way as to reduce the pile to one where the first player to choose (now B) does not have a winning strategy, then A will win. If all three of the positions to which A can reduce the pile are positions where the first player to choose (now B) has a winning strategy, then A will lose (as B can follow a winning strategy no matter how A chooses initially).

If $n = 7$, then A can remove 1, 3, or 4, leaving B with 6, 4, or 3, respectively, and all three of these possibilities have a winning strategy for the player who chooses first (here, B). Hence, B can force A to lose, so A does not have a winning strategy for $n = 7$.

We're now ready to write down a solution to the original problem. You will see that our solution will be quite short because of all of the work we've done in advance. (This is a great technique—do the legwork beforehand so that things are simpler later on.) If you don't feel like you've got a grip on the problem, then try doing some larger cases, such as $n = 8$ to $n = 12$. Then read on.

Solution: From above, A has a winning strategy if $n = 1, 3, 4, 5, 6$, but does not have a winning strategy if $n = 2$ or $n = 7$.

Starting with a pile of size n , A must reduce the pile to one of size $n - 1$, $n - 3$, or $n - 4$ and pass it to B . If the first person to choose (now B) has a winning strategy starting with a pile of each of these sizes, then A will lose. In other words, if A has a winning strategy starting with piles of size $n - 1$, $n - 3$, and $n - 4$, then A will lose starting with a pile of size n , since B can implement A 's strategy for the smaller pile and win, no matter what A does.

If one or more of these pile sizes are such that the first person does not have a winning strategy, then A should reduce to this size, which prevents B from being able to win. Thus, A herself will win.

From $n = 8$, A can reduce to 7, 5, or 4. Since the first player does not win when starting with 7, then A wins for $n = 8$ by taking 1 toothpick and reducing the pile to 7.

For $n = 9$, A can reduce the pile to 8, 6, or 5. Since the first player has

a winning strategy for each of these sizes, we see that A loses when $n = 9$.

For $n = 10$ and $n = 11$, A can reduce the pile to 7 by removing 3 and 4, respectively. Since the first player does not have a winning strategy starting with 7, then A wins starting with $n = 10$ and $n = 11$.

For $n = 12$ and $n = 13$, A can reduce the pile to 9 by removing 3 and 4, respectively. Since the first player does not have a winning strategy starting with 9, then A wins starting with $n = 12$ and $n = 13$.

Continuing to examine cases in this way, we can list the winning and losing starting positions for A :

Winning: 15, 17, 18, 19, 20, 22, 24, 25, 26, 27, 29, 31, 32, 33, 34, ...

Losing: 14, 16, 21, 23, 28, 30, 35, ...

Therefore (to answer the question that was asked!), Anne has a winning strategy when n is 31, 32, 33, and 34, and does not when n is 35.

This is an appealing problem in many ways. There are several interesting things to think about—what a winning strategy means, how winning strategies for some positions correspond to winning strategies at other positions, and so on.

There are also some interesting extensions here for those of you who like to look a little bit beyond. Can you figure out who has a winning strategy if $n = 100$? Can you determine a complete list of winning positions for the two players? What happens if the players can remove 1, 2, or 4 instead of 1, 3, or 4? How about 1, 3, or 6? There is always more to think about!