

## Pólya's Paragon

### Now You See It, Now You Don't

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I'm sure you learned the knack of *cancelling* at some point. In fact, cancelling has probably become so second-nature that you do it quite without thinking about it. For instance, in a sum like  $x^2 - 3x + 3x - 1$  or a fraction like  $\frac{49}{35}$ , eliminating the  $3x$  terms from the sum or the factor 7 from the numerator and denominator of the fraction is almost automatic.

Cancelling can be of great benefit in solving problems, but sometimes it can hide some of the structure of a problem from us. No doubt you have already encountered one way this can happen, namely, the technique of completing the square.

For example, suppose we are asked to show that the expression  $x^2 - x + \frac{1}{2}$  is positive for all real  $x$ . Completing the square *undoes* some simplifying to show that

$$x^2 - x + \frac{1}{2} = x^2 - 2\left(\frac{1}{2}x\right) + \frac{1}{4} + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{1}{4}.$$

The right side can never be less than  $1/4$ , since the square is non-negative. In this case, inserting some additional terms allows us to rewrite the expression in a way which is more appropriate to the problem.

**Problem 1.** Show that for any positive real numbers  $x$  and  $y$ ,

$$4(x^3 + y^3) \geq (x + y)^3.$$

*Solution 1:* It's tempting to expand the right side here and work with the resulting expression:

$$4(x^3 + y^3) \geq x^3 + 3x^2y + 3xy^2 + y^3.$$

This can work, but again the resulting cancelling can sometimes eliminate too much.

We'll take another approach. The important idea here is that the expression  $x^3 + y^3$  actually factors into  $(x + y)(x^2 - xy + y^2)$ . (Check that!) Using this factorization and cancelling a factor  $(x + y)$  from each side (so we've temporarily assumed that  $x \neq -y$ ), we can rewrite the inequality as

$$4(x^2 - xy + y^2) \geq (x + y)^2,$$

or, after simplifying,

$$3x^2 - 6xy + 3y^2 \geq 0.$$

Since this is equivalent to  $3(x - y)^2 \geq 0$ , we have reduced the inequality to something that must always hold. Thus, as in the first example, in a sense

we have *unsimplified* part of the expression in order to obtain our solution. [Strictly speaking of course, to finish up this problem, we need to show that these steps are all reversible, and take care of the assumption we introduced, but I will leave that for you.]

Sometimes there can be a great deal of *potential* cancelling in an expression, even if at first it is not obvious.

For example, consider the following sum:

$$\sum_{k=1}^{1000} \left( \frac{1}{k} - \frac{1}{k+1} \right).$$

It is tempting to simplify the expression in brackets by combining the two terms, but that gets us nowhere quickly; although each expression reduces to a term which is not complicated, we still need to add 1000 such terms! In this case, it is far easier to write out the sum:

$$\left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{999} - \frac{1}{1000} \right) + \left( \frac{1}{1000} - \frac{1}{1001} \right).$$

Now you can see that the two  $1/2$ s cancel, as do the two  $1/3$ s, the two  $1/4$ s, and so on, up to and including the two  $1/1000$ s. The entire sum collapses leaving only the first and last terms. So we see that

$$\sum_{k=1}^{1000} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{1001} = \frac{1000}{1001}.$$

A sum such as this is called a *telescoping sum*, since this collapsing is a little like the way the sections of a small telescope collapse into one another. We will have a similar collapse with any sum that has the form

$$\sum_{k=1}^n (F(k) - F(k+1)) \quad \text{or} \quad \sum_{k=1}^n (F(k+1) - F(k)),$$

where  $F(k)$  is some function of  $k$ . If we write this out in the longer form, the negative term in one bracket cancels with the positive term in the next.

The real power of this kind of cancellation shows itself when a more complicated expression can be rearranged into a telescoping form, as in the next example.

**Problem 2.** Compute the sum

$$\sum_{k=1}^{2007} \frac{1}{k(k+1)}.$$

This is again a large sum, but its terms certainly do not have the form  $F(k) - F(k+1)$ . Or do they? We need to put our simplification hat on backwards here (like we did earlier) and pull this term apart. We get

$$\frac{1}{k(k+1)} = \frac{(k+1) - k}{k(k+1)} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Now, proceeding as in our example above, we get

$$\sum_{k=1}^{2007} \frac{1}{k(k+1)} = \sum_{k=1}^{2007} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{2008} = \frac{2007}{2008}.$$

A similar idea applies to products. The standard notation for products in mathematics is  $\prod_{k=1}^n a_k$ , which represents the product  $a_1 a_2 a_3 \cdots a_n$ . An expression like  $\prod_{k=1}^{20} \frac{k+1}{k}$  can be written out as

$$\prod_{k=1}^{20} \frac{k+1}{k} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{20}{19} \cdot \frac{21}{20}.$$

(Note that the *last* factor in the product has  $k = 20$  in the denominator.) This time the cancelling is even easier to see, and we wind up with the answer 21.

We will have a similar collapse with any product that has the form

$$\prod_{k=1}^n \frac{F(k+1)}{F(k)} \quad \text{or} \quad \prod_{k=1}^n \frac{F(k)}{F(k+1)},$$

where, as before,  $F(k)$  is some function of  $k$ . If we write out the product in the longer form, the numerator in one bracket cancels with the denominator in the next (or vice versa).

In dealing with such sums and products, the main difficulty is often rearranging everything into the correct form.

I'll close with a few problems for you to try yourself. (The last one will require at least one trigonometric identity.) You might even look through this month's Mayhem problems too!

1. Show that for any positive integer  $n$ , the value of  $n^7 - n$  must always be a multiple of 7.

2. Find the sum

$$\frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{7}} + \cdots + \frac{1}{\sqrt{2005} + \sqrt{2007}}.$$

3. Find the product

$$\prod_{k=1}^{100} \frac{k^2 + 4k + 4}{k^2 + 3k + 2}.$$

4. Simplify the expression

$$\frac{\tan 1}{\cos 2} + \frac{\tan 2}{\cos 4} + \frac{\tan 4}{\cos 8} + \cdots + \frac{\tan 128}{\cos 256}.$$