Iterating Möbius Functions with Rational Coefficients, Part II

Kun-Chieh Wang

A Möbius function with rational coefficients is a function of the form $f(z) = \dfrac{az+b}{cz+d}$, where z is a complex variable and a, b, c, and d are rational numbers such that $ad \neq bc$. For such a function f, we consider the sequence $\{f_k\}_{k=0}^{\infty}$ of functions, where f_0 is the identity function, $f_1 = f$, and $f_k = f \circ f_{k-1}$ for $k \geq 2$. The sequence $\{f_k\}$ is said to be periodic if there exists a positive integer n such that $f_n = f_0$. The smallest such integer n is the period of the sequence.

In Part I, we found Möbius functions with rational coefficients that generate sequences with periods 1, 2, 3, 4, and 6, and we proved that periods 5, 8, and 12 are not possible. We then made a conjecture, which we now prove.

Theorem. Every periodic sequence $\{f_k\}$ generated by iterating a Möbius function f with rational coefficients has period 1, 2, 3, 4, or 6.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant of A is $\det A = ad - bc$; the *trace* of A is $\operatorname{tr} A = a + d$. The *characteristic equation* of A is $\det(A - xI) = 0$, where I is the 2×2 identity matrix. The roots of the characteristic equation are the *eigenvalues* of A.

We will need to use some well-known results from linear algebra, which we state without proof. For any two 2×2 matrices A and B, we have $\det(AB) = \det(A)\det(B)$. For any 2×2 matrix A, the eigenvalues are λ and μ if and only if $\lambda + \mu = \operatorname{tr}(A)$ and $\lambda\mu = \det(A)$. If λ and μ are the eigenvalues of a 2×2 matrix A, then λ^k and μ^k are the eigenvalues of A^k, for any positive integer k.

We will also need the following results.

Lemma 1. If there is a sequence $\{f_k\}$ of period n obtained by iterating a Möbius function f with rational coefficients, then there is such a sequence of period m for every positive divisor m of n.

Proof: Let $n = m\ell$. Suppose $A^n = \alpha I$, where A is the coefficient matrix of f and α is some non-zero rational number. Let $B = A^\ell$. Then $B^m = \alpha I$. If $B^j = \beta I$ for some $j < m$ and some rational $\beta \neq 0$, then $A^{j\ell} = \beta I$. This is not possible, since $j\ell < n$ and n is the period of $\{f_k\}$. ∎

Lemma 2. Let n be an odd positive integer. If there is a Möbius function f with rational coefficients such that iteration of f generates a sequence $\{f_k\}$ of period n, then there is such a function f with a coefficient matrix B such that $B^n = I$.

Proof: Let f be a Möbius function with rational coefficients such that iteration of f generates a sequence $\{f_k\}$ of period n, and let A be the coefficient matrix of f. Then $A^n = \alpha I$ for some rational number $\alpha \neq 0$. Note that $A^{2n} = \alpha^2 I$ and $\alpha^2 = \det(\alpha I) = \det(A^n) = (\det A)^n$. Hence, $\alpha^{2/n} = \det A$, which is a non-zero rational number. Let g be the Möbius function whose coefficient matrix is $B = \frac{1}{\alpha^{2/n}} A^2$, and let $\{g_k\}$ be the sequence obtained by iterating g. The entries of B are rational and $B^n = I$.

Now suppose that $B^k = \beta I$ for some positive divisor k of n and some rational number $\beta \neq 0$. Then $A^{2k} = \left(\alpha^{2/n}\right)^k \beta I$. Hence, $2k \geq n$. Since n is odd, $k > n/2$. Then $k = n$. We conclude that $\{g_k\}$ has period n. ■

An important tool is the following trigonometric result.

Lemma 3. If both θ/π and $\cos\theta$ are rational, then $\cos\theta \in \{0, \pm 1, \pm\frac{1}{2}\}$.

Proof: Let $\theta/\pi = m/n$ where m is an integer and n is a positive integer. Then $\cos n\theta = \cos m\pi = (-1)^m$. Trigonometric identities yield

$$2\cos n\theta \;=\; (2\cos\theta)^n + a_1(2\cos\theta)^{n-1} + \cdots + a_{n-1}(2\cos\theta) + a_n\,,$$

where the coefficients a_i are integers. Now $2\cos\theta$ is a rational root of the monic polynomial $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ with integer coefficients. Hence, $2\cos\theta$ must be an integer. It follows that $\cos\theta \in \{0, \pm 1, \pm\frac{1}{2}\}$.

Proof of Theorem: Let A be the coefficient matrix of f. Suppose $A^n = \alpha I$ for some odd positive integer n and some non-zero rational number α. By Lemma 2, we may take $\alpha = 1$. Then $(\det A)^n = \det(A^n) = \det I = 1$. Since n is odd, $\det A = 1$. Let λ and μ be the eigenvalues of A. Then λ^n and μ^n are the eigenvalues of $A^n = I$. Hence, $\lambda^n = 1$ and $\mu^n = 1$. Now $\lambda\mu = \det(A) = 1$. Since both λ and μ are n^{th} roots of unity, $\mu = \overline{\lambda}$. We also have $\lambda + \mu = \operatorname{tr} A$, which is rational.

Let $\lambda = \cos\theta + i\sin\theta$, where $\theta = 2t\pi/n$ for some integer t such that $0 \leq t < n$. Note that we have $\cos\theta = \frac{1}{2}\left(\lambda + \overline{\lambda}\right) = \frac{1}{2}\left(\lambda + \mu\right) = \frac{1}{2}\operatorname{tr} A$. This is rational, as is $\theta/\pi = 2t/n$. By Lemma 3, $\cos(2t\pi/n) \in \{0, \pm 1, \pm\frac{1}{2}\}$. Hence, the only possible odd values for n are 1 or 3.

By Lemma 1 and the result on odd values for n, the only possible even values for n must have the form 2^u or $2^u 3$ with $u \geq 1$. By Lemma 1 again, the elimination of 8 and 12 eliminates all but 2, 4, and 6 as possible even values for n. This completes the proof of the theorem. ■

Acknowledgement: The author thanks Dr. I.E. Leonard of Canada for some helpful information on linear algebra.

Kun-Chieh Wang
Cheng-Kuo High School
Taipei
TAIWAN
wkc751204@yahoo.com.tw