SKOLIAD No. 85

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 July, 2005***. A copy of MATHEMATICAL MAYHEM Vol. 3 will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (*) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

Our items this issue come from the 4th annual CNU Regional Mathematics Contest. Only the first 13 Questions have been included; the rest will appear in forthcoming Skoliads. Thanks go to R. Porsky, C.N.U., Newport News, VA.

4th Annual CNU Regional High School Mathematics Contest Saturday December 6, 2003

1. (*) If 64 is divided into three parts proportional to 2, 4, and 6, the smallest part is:

(A) $5\frac{1}{3}$ (B) 11 (C) $10\frac{2}{3}$ (D) none of these

2. (*) If, in applying the quadratic formula to a quadratic equation $f(x) = ax^2 + bx + c = 0$, it happens that $c = \dfrac{b^2}{4a}$, then the graph of $y = f(x)$ will certainly:

(A) have a maximum (B) have a minimum
(C) be tangent to the x–axis (D) be tangent to the y–axis

3. (*) Let $\{a_n\}$ be a geometric sequence. If $a_1 = 8$ and $a_7 = 5832$, then a_5 is:

(A) 648 (B) 832 (C) 1168 (D) 1944

4. (*) The area enclosed by $|x| + |y| = 1$ is:

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

5. $(*)$ If the graph of $f(x) = \big||x-2|-a\big| - 3$ has exactly three x–intercepts, then a equals:

(A) 3 (B) 4 (C) 0 (D) -3

6. $(*)$ If $\frac{m}{n} = \frac{4}{3}$ and $\frac{r}{t} = \frac{9}{14}$, then the value of $\dfrac{3mr - nt}{4nt - 7mr}$ is:

(A) $-5\frac{1}{2}$ (B) $-\frac{11}{14}$ (C) $-\frac{2}{3}$ (D) $-1\frac{1}{4}$

7. $(*)$ Which functions satisfy $f\left(\dfrac{x+y}{2}\right) = \frac{1}{2}(f(x) + f(y))$?

(A) $\ln x$ (B) $\dfrac{1}{x}$ (C) $2x$ (D) 2^x

8. $(*)$ Let x, $y > 0$, $x > y$, and $z \neq 0$. The inequality which is not always correct is:

(A) $x + z > y + z$ (B) $x - z > y - z$
(C) $xz > yz$ (D) $xz^2 > yz^2$

9. $(*)$ If $a^x = c^q = b$ and $c^y = a^z = d$, then:

(A) $xy = qz$ (B) $x + y = q + z$ (C) $x - y = q - z$ (D) $x^y = q^z$

10. $(*)$ The area of a square inscribed in a semicircle is to the area of the square inscribed in the entire circle as:

(A) $1:2$ (B) $2:3$ (C) $2:5$ (D) $3:4$

11. $(*)$ If $0 < \alpha, \beta < \frac{\pi}{2}$ and $\alpha > \beta$, then:

(A) $\sin(\alpha - \beta) > \sin\alpha - \sin\beta$ (B) $\sin(\alpha - \beta) < \sin\alpha - \sin\beta$
(C) $\sin(\alpha - \beta) = \sin\alpha - \sin\beta$ (D) none of these

12. $(*)$ Let $f(x) = 3^x + 5$. Then the domain of f^{-1} is:

(A) $(0, +\infty)$ (B) $(5, +\infty)$ (C) $(8, +\infty)$ (D) $(-\infty, +\infty)$

13. $(*)$ A man has a pocket full of change, but can not make change for a dollar. What is the greatest value of coins he could have?

(A) $\$0.99$ (B) $\$1.09$ (C) $\$1.19$ (D) $\$1.29$

4^e Concours Annuel CNU Régional de Mathématique du Secondaire Samedi, le 6 Décembre 2003

1. $(*)$ Si 64 est divisé en 3 parties proportionnelles à 2, 4 et 6, la plus petite de ces parties est :

(A) $5\frac{1}{3}$ (B) 11 (C) $10\frac{2}{3}$ (D) aucune d'elles

2. $(*)$ Si, en appliquant la formule quadratique à l'équation $f(x) = ax^2 + bx + c = 0$, on obtient $c = \dfrac{b^2}{4a}$, donc le graphique de $y = f(x)$ va certainement :

(A) avoir un maximum (B) avoir un minimum
(C) être tangent a l'axe des x (D) être tangent a l'axe des y

3. $(*)$ Soit $\{a_n\}$ une suite géometrique. Si $a_1 = 8$ et $a_7 = 5832$, alors a_5 vaut :

(A) 648 (B) 832 (C) 1168 (D) 1944

4. $(*)$ L'aire comprise dans $|x| + |y| = 1$ est :

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

5. $(*)$ Si le graphique de $f(x) = \big||x-2| - a\big| - 3$ a 3 zéros exactement, alors a vaut :

(A) 3 (B) 4 (C) 0 (D) -3

6. $(*)$ Si $\dfrac{m}{n} = \dfrac{4}{3}$ et $\dfrac{r}{t} = \dfrac{9}{14}$, alors la valeur de $\dfrac{3mr - nt}{4nt - 7mr}$ est :

(A) $-5\frac{1}{2}$ (B) $-\frac{11}{14}$ (C) $-\frac{2}{3}$ (D) $-1\frac{1}{4}$

7. $(*)$ Quelles fonctions satisfont $f\left(\dfrac{x+y}{2}\right) = \dfrac{1}{2}(f(x) + f(y))$?

(A) $\ln x$ (B) $\dfrac{1}{x}$ (C) $2x$ (D) 2^x

8. $(*)$ Soit x, $y > 0$, $x > y$ et $z \neq 0$. L'inégalité qui n'est pas toujours correcte est :

(A) $x + z > y + z$ (B) $x - z > y - z$
(C) $xz > yz$ (D) $xz^2 > yz^2$

9. (∗) Si $a^x = c^q = b$ et $c^y = a^z = d$, alors :

(A) $xy = qz$ (B) $x + y = q + z$ (C) $x - y = q - z$ (D) $x^y = q^z$

10. (∗) L'aire d'un carré inscrit dans un demi-cercle est à l'aire du carré inscrit dans le cercle en proportion :

(A) $1 : 2$ (B) $2 : 3$ (C) $2 : 5$ (D) $3 : 4$

11. (∗) Si $0 < \alpha, \beta < \frac{\pi}{2}$ et $\alpha > \beta$, alors :

(A) $\sin(\alpha - \beta) > \sin\alpha - \sin\beta$ (B) $\sin(\alpha - \beta) < \sin\alpha - \sin\beta$
(C) $\sin(\alpha - \beta) = \sin\alpha - \sin\beta$ (D) aucune d'elles

12. (∗) Soit $f(x) = 3^x + 5$. Alors le domaine de f^{-1} est :

(A) $(0, +\infty)$ (B) $(5, +\infty)$ (C) $(8, +\infty)$ (D) $(-\infty, +\infty)$

13. (∗) Un homme a une poche pleine de change, mais ne peut pas faire de change pour un dollar. Quelle est la plus grande valeur possible de son change ?

(A) $0,99$$ (B) $1,09$$ (C) $1,19$$ (D) $1,29$$

Next we give the solutions to the 2004 BC Colleges High School Mathematics Contest, Final Round – Part B, for both Junior and Senior levels [2004 : 385–387].

2004 BC Colleges High School Mathematics Contest
Junior Final Round – Part B

1. The numbers greater than 1 are arranged in an array, in which the columns are numbered 1 to 5 from left to right, as shown:

(1)	(2)	(3)	(4)	(5)
	2	3	4	5
9	8	7	6	
	10	11	12	13
17	16	15	14	
⋮	⋮	⋮	⋮	⋮

(a) In which column will 2004 fall?

(b) In which column will 1999 fall?

(c) In which column(s) could $n^2 + 1$ fall, where n is a positive integer?

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Modulo 8, we see that residue 1 belongs to column (1); residues 0 and 2 belong to (2); 3 and 7 belong to (3); 4 and 6 belong to (4); and 5 belongs to (5). Now $2004 \equiv 4 \pmod 8$ and $1999 \equiv 7 \pmod 8$. Thus, 2004 and 1999 will fall in columns (4) and (3), respectively. Now n^2 is congruent modulo 8 to one of 0, 1, or 4. Hence, $n^2 + 1$ could appear only in columns (1), (2), or (5).

2. How many sets of two or more consecutive positive integers have a sum of 105?

Official solution.

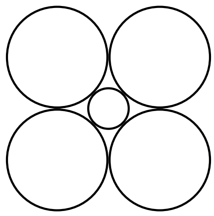
If we have an odd number of consecutive integers with a sum of 105, such as $34 + 35 + 36 = 105$, we see that the middle integer is their average: $\frac{34+35+36}{3} = \frac{105}{3} = 35$. This means that the sum (105) divided by the odd number (of consecutive integers) must be an integer. Since $105 = 3 \times 5 \times 7$, the divisors of 105 are 1, 3, 5, 7, 15, 21, 35, and 105 (all of which are odd). We eliminate 1 because we need at least two consecutive integers. On the other hand, if we had 15 consecutive integers, then, since $\frac{105}{15} = 7$, the 15 consecutive integers would have to be 0, 1, ..., 14. But not all of these are positive. In this way we eliminate the cases of 15, 21, 35, and 105 consecutive integers, leaving 3, 5, and 7.

If we have the sum of an even number of consecutive integers, such as $52 + 53 = 105$, we see that the average is midway between the middle two terms. In other words, the average of the integers must be an integer plus $\frac{1}{2}$. Consequently, 105 divided by half the number of terms must be an integer. We have the same divisors as before, namely 1, 3, 5, 7, 15, 21, 35, and 105. Thus, we could have 2, 6, 10, 14, 30, 42, 70, or 210 consecutive integers adding up to 105. The requirement that all the integers be positive leads us to eliminate anything greater than 14.

Therefore, in total there are seven ways to write 105 as a sum of at least two consecutive, positive integers:

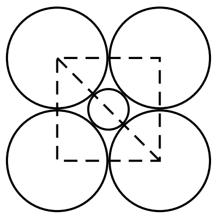
number	average	sum
2	52.5	$52 + 53$
3	35	$34 + 35 + 36$
5	21	$19 + 20 + 21 + 22 + 23$
6	17.5	$15 + 16 + 17 + 18 + 19 + 20$
7	15	$12 + 13 + 14 + 15 + 16 + 17 + 18$
10	10.5	$6 + 7 + \cdots + 10 + 11 + \cdots + 14 + 15$
14	7.5	$1 + 2 + \cdots + 6 + 7 + 8 + 9 + \cdots + 13 + 14$

3. The centres of four circles of radius 12 form a square. Each circle is tangent to the two circles whose centres are the vertices of the square that are adjacent to the centre of the circle. A smaller circle, with centre at the intersection of the diagonals of the square, is tangent to each of the four larger circles. Find the radius of the smaller circle.



Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Clearly, the main diagonal of the square formed by the centres of the four circles of radius 12 has length $24\sqrt{2}$. But it is also twice the radius of the small circle plus twice the radius of a large circle, because the points of tangency of the circles must be on the main diagonal (by symmetry). Thus, $r = 12(\sqrt{2} - 1)$ immediately.



4. The Fibonacci Sequence begins: $1, 1, 2, 3, 5, 8, 13, 21, \ldots$. (Each number beyond the second number is the sum of the previous two numbers.) The notation f_n means the n^{th} number; for example, $f_4 = 3$ and $f_7 = 13$.

(a) Which of the following terms in the Fibonacci Sequence are odd? Explain your conclusions.

$$f_{38}\,,\quad f_{51}\,,\quad f_{150}\,,\quad f_{200}\,,\quad f_{300}$$

(b) Which of the following terms in the Fibonacci Sequence are divisible by 3? Explain your conclusions.

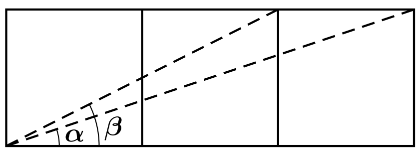
$$f_{48}\,,\quad f_{75}\,,\quad f_{196}\,,\quad f_{379}\,,\quad f_{1000}$$

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

(a) Modulo 2, the Fibonnaci Sequence is $1, 1, 0, 1, 1, 0, \overline{1, 1, 0}, \ldots$. Clearly, this pattern goes on forever, since each residue is determined by the two residues before it. Hence, every 3^{rd} term is even, namely f_{51}, f_{150}, and f_{300} (and the rest are odd, f_{38} and f_{200}).

(b) Modulo 3, the sequence is $\overline{1, 1, 2, 0, 2, 2, 1, 0}, \ldots$. Thus, every 4^{th} term is divisible by 3, namely f_{48}, f_{196}, f_{1000}.

5. The diagram shows three squares. Find the measure of the angle $\alpha + \beta$.



I. *First solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

We have $\tan\alpha = 1/3$ and $\tan\beta = 1/2$. Thus,

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{1/3+1/2}{1-1/6} = \frac{5/6}{5/6} = 1\,.$$

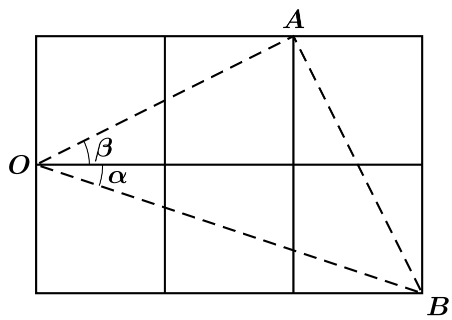
Since α and β belong to $[0,\pi/2]$, and since tan is a single-valued function on the interval $[0,\pi]$ (excluding $\pi/2$, of course), it follows that $\alpha+\beta$ is $\pi/4$.

II. *Second solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

Let $O=(0,0)$, $A=(2,1)$, and $B=(3,-1)$. [*Ed:* See the diagram in the official solution below.] One can see that $AO = AB$, because both are diagonals of a 2×1 rectangle. A rotation of 90° clockwise centred at A maps B to O. Thus, $\angle OAB = 90^\circ$. Now, $\angle AOB$ must be 45° (since $\triangle AOB$ is isoceles); that is, $\alpha+\beta = 45^\circ$.

III. *Official solution.*

If we flip the squares as shown in the diagram, we get a triangle with $\alpha+\beta = \angle AOB$. We can use the Pythagorean Theorem to calculate $OA = AB = \sqrt{5}$ and $OB = \sqrt{10}$. Then $OB^2 = OA^2 + AB^2$, implying that $\triangle AOB$ is a right triangle with $\angle OAB = 90^\circ$. Since the triangle is also isosceles, we see that



$$\angle AOB = \alpha+\beta = 45^\circ\,.$$

2004 BC Colleges High School Mathematics Contest
Senior Final Round – Part B

1. Find the number of different 7–digit numbers that can be made by rearranging the digits in the number 3053345.

Official solution.

Suppose that we give the three 3s and two 5s different colours so that we can distinguish them. Since a valid number cannot start with a 0, there are 6(6!) ways to arrange the seven digits into a valid number. In each of these, there are 3! ways to arrange the three 3s and 2! ways to arrange the two 5s. Since the 3s and 5s are indistinguishable without the colours, the number of 7–digit numbers that can be made is $\frac{6(6!)}{3!2!} = 6\cdot 5\cdot 4\cdot 3 = 360$.

Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON. There was one incorrect solution submitted.

2. The number 2004 has only 12 integer factors, including 1 and 2004.

(a) How many distinct factors does 2004^4 have?

(b) If the product of the factors in part (a) is written as 2004^N, find the value of N?

Solution to part (a) *by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

Since $2004 = 167 \cdot 3 \cdot 2^2$, we see that $2004^4 = 167^4 \cdot 3^4 \cdot 2^8$. Thus, 2004^4 has $(4+1)(4+1)(8+1) = 225$ distinct factors.

Official solution to part (b).

The product of the 225 factors from part (a) will be of the form $2^\alpha \times 3^\beta \times 167^\gamma$, for some positive integers α, β, and γ. We need to find these integers.

Each factor of 2004^4 will include exactly one of 2^0, 2^1, $\ldots$, 2^8. Multiplying these together gives $2^{(0+1+2+\cdots+8)} = 2^{36}$. Each of these possible factors occurs once for each possible combination of the powers on 3 and 167. Since there are 5 possible powers for both 5 and 167, this gives 25 combinations. Thus, the product of the factors of 2004^4 contains $2^{(36\times 25)}$; that is, $\alpha = 36 \times 25 = 900$.

Similarly, multiplying the possible powers of 3 gives $3^{(0+1+\cdots+4)} = 3^{10}$ and, with the 9 possible powers for 2 and the 5 possible for 167, there are 45 possible combinations giving this factor. Hence, 2004^4 contains $3^{(10\times 45)}$; that is, $\beta = 10 \times 45 = 450$. The same argument shows that $\gamma = 450$. Thus, the product of all of the factors of 2004^4 is $2^{900} \times 3^{45} \times 167^{450} = 2004^{450}$.

Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON. Part (b) was also solved by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

3. You are given two parallel panes of glass. Each pane will transmit 70%, reflect 20%, and absorb 10% of the light that falls on it. For example, for the portion of a beam of light incident on the pane on the left that follows the path in Figure 1, the fraction transmitted is $0.7 \times 0.7 = 0.49$, but for the portion of the beam following the path shown in Figure 2, the fraction transmitted is $0.7 \times 0.2 \times 0.2 \times 0.7 = 0.0196$. If a light source is placed on one side of the two panes, find the total fraction of light that passes through to the other side.

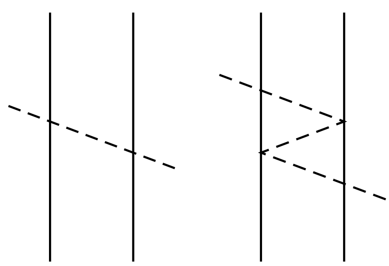


Figure 1 Figure 2

Official solution.

If light is seen through the two panes of glass, it will have been either unreflected, or reflected twice, four times, six times, etc. Since 70% of the light is transmitted through each pane and 20% is reflected, the total fraction of light that passes through to the other side is

$$0.7 \times 0.7 \;+\; 0.7 \times 0.2^2 \times 0.7 \;+\; 0.7 \times 0.2^4 \times 0.7 \;+\; \cdots$$
$$= \;0.7^2 \times (1 + 0.2^2 + 0.2^4 + \cdots) \;=\; 0.49 \times \frac{1}{1 - 0.04} \;=\; \frac{49}{96}\,.$$

There was one incorrect solution submitted.

4. Let f be a function whose domain is all real numbers. If

$$f(x) + 2f\left(\frac{x+2001}{x-1}\right) \;=\; 4013 - x$$

for all x not equal to 1, find the value of $f(2003)$.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Consider $x = 2$ and $x = 2003$ in the given equation. We now have

$$\begin{aligned} f(2) + 2f(2003) &= 4011\,, \\ f(2003) + 2f(2) &= 2010\,. \end{aligned}$$

Solving this system yields $f(2) = 3$ and $f(2003) = 2004$.

Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON.

5. An ant is crawling at a rate of 48 centimetres per minute along a strip of rubber which can be infinitely and uniformly stretched. The strip is initially one metre long and one centimetre wide and is stretched an additional one metre at the end of each minute. Assume that when the strip is stretched, the ratio of the distances from each end of the strip remains the same before and after the stretch. If the ant starts at one end of the strip of rubber, find the number of minutes until it reaches the other end.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

At the end of the first minute, the ant has traveled 48 percent of the way. At the end of the second minute the ant has travelled $48 + 48/2$ percent of the way. After the 3^{rd} minute, $48(1 + 1/2 + 1/3)$ percent; after the 4^{th} minute, $48(1 + 1/2 + 1/3 + 1/4)$ percent, which is exactly 100 percent. Clearly, the function mapping minutes to percentage is strictly increasing. Thus, after only 4 minutes, the ant reaches the end.

That brings us to the end of another issue. This months winner of a past volume of Mayhem is Alex Wice. Congratulations, Alex! Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Amy Cameron (Carleton University), Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Problems

Please send your solutions to the problems in this edition by **1 September 2005**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M188. *Proposed by Charalampos Stergiou, Chalkida, Greece.*

Consider triangle ABC in which $\angle B = \angle C = 35^\circ$. In the interior of the triangle we take a point M such that $\angle MBC = 25^\circ$ and $\angle MCB = 30^\circ$. Prove, without trigonometry, that $\angle AMC = 150^\circ$.

M189. *Proposed by Mihály Bencze, Brasov, Romania.*

Find all real solutions of the following system of equations:

$$\begin{aligned} x + \sqrt{x^2+1} &= 10^{y-x}\,, \\ y + \sqrt{y^2+1} &= 10^{z-y}\,, \\ z + \sqrt{z^2+1} &= 10^{x-z}\,. \end{aligned}$$

M190. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Given any three points in a unit square, show that a pair of them must be no further apart than $\sqrt{6}-\sqrt{2}$.

M191. *Proposed by the Mayhem Staff.*

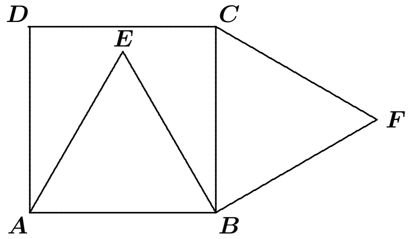
The surface areas of the six faces of a rectangular prism (box) are 1254, 1254, 770, 770, 1995, and 1995 cm^2. Determine the volume of the prism.

M192. *Proposed by Victor Oxman, Western Galilee College, Israel.*

In triangles $A_1B_1C_1$ and $A_2B_2C_2$, we are given that $A_1C_1 = A_2C_2$, that the medians B_1M_1 and B_2M_2 are equal, and that the bisectors A_1D_1 and A_2D_2 are equal. Prove that the triangles are congruent.

M193. *Proposed by Robert Bilinski, Outremont, QC.*

On square $ABCD$, an equilateral triangle ABE is constructed internally and an equilateral triangle BCF is constructed externally. Prove that the points D, E, and F are collinear.



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M188. *Proposé par Charalampos Stergiou, Chalkida, Grèce.*

Dans un triangle ABC les angles B et C mesurent $35°$. On choisit un point M à l'intérieur du triangle de sorte que les angles MBC et MCB valent respectivement $25°$ et $30°$. Sans trigonométrie, montrer que l'angle AMC vaut $150°$.

M189. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Trouver toutes les solutions réelles du système d'équations suivant :

$$\begin{aligned} x+\sqrt{x^2+1} &= 10^{y-x}\,, \\ y+\sqrt{y^2+1} &= 10^{z-y}\,, \\ z+\sqrt{z^2+1} &= 10^{x-z}\,. \end{aligned}$$

M190. *Proposé par Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Etant donné trois points dans un carré unité, montrer qu'une paire d'entre eux ne peuvent être distants de plus de $\sqrt{6}-\sqrt{2}$.

M191. *Proposé par Équipe de Mayhem.*

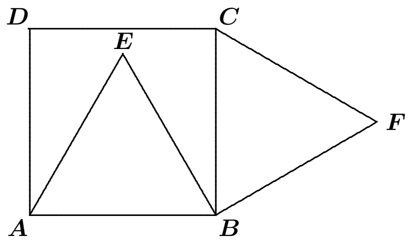
Les aires des six faces d'un prisme rectangulaire (une boîte) valent 1254, 1254, 770, 770, 1995 et 1995 cm^2. Calculer le volume du prisme.

M192. *Proposé par Victor Oxman, Western Galilee College, Israël.*

Montrer que les triangles $A_1B_1C_1$ et $A_2B_2C_2$ sont congruents, sachant que $A_1C_1 = A_2C_2$, que les médianes B_1M_1 et B_2M_2 ainsi que les bissectrices A_1D_1 et A_2D_2 sont égales.

M193. *Proposé par Robert Bilinski, Outremont, QC.*

On trace à l'intérieur du carré $ABCD$ un triangle équilatéral ABE et à l'extérieur un triangle équilatéral BCF. Montrer que les points D, E et F sont alignés.



Mayhem Solutions

M130. *Proposed by the Mayhem Staff.*

Tickets are numbered $1, 2, 3, 4, \ldots, N$. Exactly half of the tickets have the digit 1 on them.

(a) If N is a three-digit number, determine all possible values of N.

(b) Determine some possible values for N if N is a four-digit number, or a five-digit number, etc.

Solution by Doug Newman, Lancaster, CA, USA, modified by the editor.

(a) We begin by counting the numbers less than 100 that contain the digit 1. Among the nine single-digit numbers, only one contains 1; among the two-digit numbers, those from 10 to 19 each contain 1 in the tens position, while there are only 8 further two-digit numbers which contain 1, namely $21, 31, \ldots, 91$. Thus, there are 19 numbers less than 100 that contain 1.

From 100 to 199, every number contains 1 in the hundreds position. Thus, from 100 to 199, there are 100 numbers containing 1. From 200 to 299, we have a repetition of the scenario from 1 to 99; that is, exactly 19 of the numbers contain 1. Similarly, in each of the ranges 300–399, 400–499, …, 900–999, exactly 19 numbers contain 1.

Let N be the number of tickets, and let T be the number of tickets which contain the digit 1. We want to find N and T such that $T = N/2$, or, equivalently, $T/N = \frac{1}{2}$. From above, the ratio T/N is approximately 0.192 when $N = 99$; it is approximately 0.598 when $N = 199$; and it is approximately 0.462 when $N = 299$. Since the growth rate of T is only greater than $\frac{1}{2}$ in the first 20 numbers of each set of 100, we can rule out any higher values of N by simply considering $N = 319$. For $N = 319$, the number of tickets containing the digit 1 is $19 + 100 + 19 + 11 = 149$. Then $T/N = 149/319 < \frac{1}{2}$. Thus, if $T/N = \frac{1}{2}$, then $100 \leq N \leq 299$.

When $100 \leq N \leq 199$, we note that T is increased by 1 each time N is increased by 1 (since all the numbers in this range have a 1 in the hundreds position). Therefore, we are seeking an integer n such that

$$\frac{T}{N} = \frac{20+n}{100+n} = \frac{1}{2}\,.$$

Solving, we find that $n = 60$, which yields $N = 100 + 60 = 160$ and $T = 20 + 60 = 80$. Hence, 160 is one of our answers.

If $200 \le N \le 300$, then increases in T are not proportional to increases in N. Let us first consider the range $200 \le N \le 209$. Then $T = 119$ for $N = 200$, and $T = 120$ for $201 \le N \le 209$. For all such values of N, we have $T/N > \frac{1}{2}$. Next consider the range $210 \le N \le 219$. In this range, the ratio T/N increases, since T increases by the same amount as N, which means that we still have $T/N > \frac{1}{2}$.

For $N \ge 219$ we have $T \ge 130$, which means that we need $N \ge 260$ in order to get $T/N = \frac{1}{2}$. However, when $N = 260$, we find that $T = 134$. Thus, we need $N \ge 268$, at which point $T = 135$, further refining N to be at least 270. Indeed, $N = 270$ is a solution. Since N must be even, the next possibility is $N = 272$, in which case $T = 136$, which means that $N = 272$ is also a solution. Now, the next number to contain the digit 1 is 281, at which point $T = 137$, which has $T/N < 0.488$. Since the value of T/N will continue to decline from this point on, we must have found all the solutions.

In summary, the solution set for N is $\{160, 270, 272\}$.

(b) By first finding the values of T for different ranges of N (see the table below) and then using the above methodology, one can determine that $N \in \{1458, 3398, 13120, 44686\}$.

From	To	# with "1s"	Cumulative Total
1	9	1	1
10	19	10	11
20	99	8	19
100	199	100	119
200	999	152 [= 8(19)]	271
1 000	1 999	1 000	1 271
2 000	9 999	2 168 [= 8(271)]	3 439
10 000	19 999	10 000	13 439
20 000	99 999	27 512 [= 8(3439)]	40 593

A solution where T only counted numbers with exactly one digit 1 was submitted by Robert Bilinski, Outremont, QC.

M131. *Proposed by the Mayhem Staff.*

The triangular array of numbers shown has the following properties:

1. The bottom row contains each of the numbers 1, 2, ..., 8 exactly once.
2. Each number in a row above the bottom row is the sum of the two neighbouring numbers in the row immediately below, if this sum is less than 10; otherwise, 9 is subtracted from this sum.

6
7 8
3 4 4
8 4 9 4
7 1 3 6 7
4 3 7 5 1 6
5 8 4 3 2 8 7
2 3 5 8 4 7 1 6

Is it possible to create a triangular array with the above properties using each number from 1 to 9 exactly four times?

Solution by Doug Newman, Lancaster, CA, USA.

The simple answer is yes. Through trial and error one can find the following two triangular arrays that satisfy the required conditions.

9 4 5 4 9 5 3 1 8 6 2 1 9 8 7 8 3 7 2 6 1 3 5 7 9 2 4 6 1 2 3 4 5 6 7 8	9 8 1 8 9 1 6 2 7 3 4 2 9 7 5 7 6 5 4 3 2 6 1 5 9 4 8 3 2 4 6 8 1 3 5 7

By recognizing that the arithmetic is performed modulo 9 and by noting certain relations that hold on the base row of numbers, such as

$$a_1 + a_8 = a_2 + a_7 = a_3 + a_6 = a_4 + a_5 = 9,$$
$$a_3,\, a_5 \in \{3,\, 6\},$$
$$a_2 + a_3 + a_4 = a_5 + a_6 + a_7 = 9,$$
$$a_1 + a_2 = a_3 \quad \text{and} \quad a_7 + a_8 = a_6,$$

one can create other such triangular arrays:

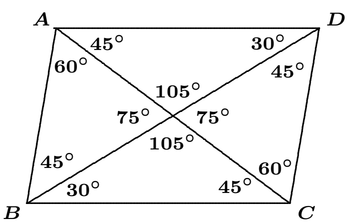
9 4 5 4 9 5 6 7 2 3 8 7 9 2 1 2 6 1 8 3 7 6 5 1 9 8 4 3 7 8 6 4 5 3 1 2	9 8 1 8 9 1 3 5 4 6 7 5 9 4 2 4 3 2 7 6 5 3 1 2 9 7 8 6 5 7 3 8 1 6 2 4
9 2 7 2 9 7 6 5 4 3 1 5 9 4 8 4 6 8 1 2 3 6 7 8 9 1 2 3 5 1 6 2 7 3 8 4	9 2 7 2 9 7 3 8 1 6 4 8 9 1 5 1 3 5 4 6 8 3 7 5 9 4 2 6 8 4 3 2 7 6 5 1

Also solved by Robert Bilinski, Outremont, QC.

M132. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

(a) What is wrong with the diagram of the parallelogram? The angle measures are in degrees.

(b) Suggest a modification that would make the diagram plausible. You are not allowed to modify the three lines through B.



Solution by Zhao Xin Hao, student, and Luyun Zhong-Qiao, Columbia International College, Hamilton, ON.

(a) Let E be the intersection of the of the diagonals AC and BD. Let $a = AE = EC$ and let $b = BE = ED$. The Law of Sines applied to $\triangle ABE$ gives us

$$\frac{b}{a} = \frac{\sin 60^\circ}{\sin 45^\circ} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}.$$

Applying the Law of Sines to $\triangle BCE$ yields

$$\frac{b}{a} = \frac{\sin 45^\circ}{\sin 30^\circ} = \sqrt{2}.$$

Since $\sqrt{2} \neq \frac{1}{2}\sqrt{6}$, the diagram is flawed.

(b) If we are to keep the three lines through B fixed, then the angles between must also be fixed. As above, we let E be the intersection of the diagonals AC and BD, and let $a = BE = EC$ and $b = BE = ED$. If we set $\alpha = \angle BAC$ and $\beta = \angle BCA$, then, by applying the Law of Sines to $\triangle ABE$ and $\triangle CBE$, we have

$$\frac{\sin 45^\circ}{\sin\alpha} = \frac{a}{b} = \frac{\sin 30^\circ}{\sin\beta}.$$

Thus,

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin 45^\circ}{\sin 30^\circ} = \frac{\sqrt{2}/2}{1/2} = \sqrt{2}.$$

In $\triangle ABC$, we have $\alpha + \beta + 75^\circ = 180^\circ$. Hence, $\alpha = 105^\circ - \beta$, and $\sin\alpha = \sin(105^\circ - \beta) = \sin 105^\circ \cos\beta - \cos 106^\circ \sin\beta$. Then

$$\begin{aligned}\sqrt{2} &= \frac{\sin 105^\circ \cos\beta - \cos 106^\circ \sin\beta}{\sin\beta} \\ &= \sin 105^\circ \cot\beta - \cos 105^\circ \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\cot\beta - \frac{\sqrt{2}-\sqrt{6}}{4}.\end{aligned}$$

(We have calculated $\sin 105^\circ$ and $\cos 105^\circ$ by noting that $105^\circ = 60^\circ + 45^\circ$, and applying the addition formulas.) Solving the above for $\cot\beta$ yields

$$\begin{aligned}\cot\beta &= \frac{\frac{4\sqrt{2}+\sqrt{2}-\sqrt{6}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{5\sqrt{2}-\sqrt{6}}{\sqrt{6}+\sqrt{2}} \\ &= \frac{(5\sqrt{2}-\sqrt{6})(\sqrt{6}-\sqrt{2})}{4} = \frac{12\sqrt{3}-16}{4} = 3\sqrt{3}-4.\end{aligned}$$

Therefore, $\beta = \cot^{-1}(3\sqrt{3}-4)$, which is approximately 39.896° (instead of 45°, as shown in the diagram). All the remaining angles can be determined from β.

Problem of the Month

Ian VanderBurgh, University of Waterloo

Problem (1997-1998 Scottish Mathematical Challenges) Tim organized a bus trip to the seaside. Initially, more than twenty of his friends said they would go on the outing. Tim calculated the individual cost by dividing the total cost by the number of participants, and was pleased to find that it was a whole number of dollars each. He announced the cost and four people dropped out. He recalculated the individual cost from the same total cost and started to collect the money. All went well until the last two people, who now said they couldn't come. On the day of the trip, Tim had to collect another $3 from each of the remaining participants. They all had a splendid day out, including the bus driver. How much did it cost each of the participants?

Have you ever tried to organize a trip before? If so, you and Tim have likely had similar experiences.

I really enjoy the entertaining problem style from these Scottish Mathematical Challenges. This particular problem has been "translated" a bit from its original format, with pounds replaced by dollars and "bus" replacing "coach".

This problem is similar to problems that we all saw when we first started learning algebra, but it turns out to be a fair bit more complicated than it first appears.

Solution. Let T be the total cost of the trip, and let N be number of people (including Tim) who still agreed to go after the initial price was announced. This means that $N+4$ people initially said they would go on the trip, where $N+4$ is at least 22 (that is, N is at least 18), and that $N-2$ people finally went on the trip.

When there were N people going on the trip, the individual price was T/N. After the last two people dropped out, the individual price became $T/(N-2)$.

From the given information, we have $\frac{T}{N}+3=\frac{T}{N-2}$; that is,

$$\begin{aligned} T(N-2)+3N(N-2) &= TN\,, \\ 3N^2-6N &= 2T\,. \end{aligned}$$

Since T must be a whole number (the initial cost per person was a whole number of dollars), the right side is even. Hence, the left side is even. Since $6N$ is even, we see that $3N^2$ must also be even, implying that N is even.

We set $N=2n$ where n is an integer (and n is at least 9, since N is at least 18), and we see that $2T=3(2n)^2-6(2n)=12n^2-12n$, or $T=6n^2-6n$.

It is now a bit tricky to figure out where to go. What is the crucial piece of information that we have yet to use to its fullest extent? The missing link is not the colour of the bus, but rather that the initial cost per person was a whole number of dollars; that is,

$$\frac{T}{N+4} = \frac{6n^2-6n}{2n+4} = \frac{3n^2-3n}{n+2}$$

is a whole number. Maybe this will help!

When we have a rational expression like this one, it is often helpful to "long-divide" the denominator into the numerator—you will see why in a minute! If you know how to long-divide polynomials, great; if not, after a bit of fiddling around, you can figure out that $3n^2-3n = (3n-9)(n+2)+18$, which implies that $\dfrac{3n^2-3n}{n+2} = 3n-9+\dfrac{18}{n+2}$.

Where do we go from here? Since $\dfrac{3n^2-3n}{n+2}$ is a whole number, we see that $3n-9+\dfrac{18}{n+2}$ is a whole number. We already know that $3n-9$ is a whole number; thus, $\dfrac{18}{n+2}$ is a whole number. Then $n+2$ must be a divisor of 18. But n is at least 9. Hence, $n+2$ is at least 11. Thus, $n+2$ must be 18, since it must be a divisor of 18. Therefore, $n=16$. Hence, $N=32$ and $T=6n^2-6n=1440$.

At this point, we have to remember what it was that we were originally asked! We need to know how much it cost each of the participants. We find this cost by calculating $\dfrac{T}{N-2} = \dfrac{1440}{30} = 48$. This means that the final cost for each of the participants was \$48.

We should go back at this stage to check our information: we have a total cost of \$1440, along with 32 participants who agreed to go after finding out the initial cost. Thus, there were 36 who initially agreed to go, giving an individual cost of $\$1440 \div 36 = \$40$, and 32 who agreed to go after learning the price, giving an individual cost of $\$1440 \div 32 = \$45$. Finally, there were 30 who went, giving the final individual cost of $\$1440 \div 30 = \$48$. All of this information agrees with what we were given.

What is the moral of this story? If you're going to organize a trip to the seaside, make sure you hone up on your algebraic skills first! You never know when they will come in handy. And make sure that the bus driver has a good time too!

Pólya's Paragon

Fun With Numbers (Part 3)

Shawn Godin

Last time I left you with the task of looking for patterns in the following table:

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49
50	51	52	53	54	55	56
57	58	59	60	61	62	63
64	65	66	67	68	69	70
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Notice the interesting pattern that shows up, each column just forms an arithmetic progression with common difference 7. Thus, if two numbers are in the same column, they must differ by a multiple of 7. They must also have the same remainder when you divide them by 7. Mathematicians say that each column in the table forms an *equivalence class*. The numbers in any one column are *equivalent* in that they yield the same remainder when you divide them by 7.

Calculate each of the following and try to see how they are related.

$$3+5\,, \qquad 10+19\,, \qquad 31+12\,, \qquad 45+20\,.$$

I hope you have come up with some ideas. In each case, we were adding a number from column 3 to a number from column 5. We ended up with a number in column 1. We can easily justify this by noticing that each pair of numbers can be written as $7a+3$ and $7b+5$ for some integers a and b. Their sum is

$$(7a+3)+(7b+5) \;=\; 7(a+b+1)+1\,.$$

Similar things happen when you look at subtraction and multiplication. (You should check this out yourself.)

Mathematicians, being extremely lazy beasts, are always looking for a short way of writing things. To show that the numbers 33 and 5 are in the same equivalence class, they write

$$33 \;\equiv\; 5 \pmod 7\,.$$

We read this as "33 is congruent to 5 modulo 7". What this means is that if we divide 33 and 5 by 7, we get the same remainder, or (equivalently) 33 and 5 differ by a multiple of 7.

We have been investigating some of the basic properties of congruences. We can write them down as follows.

Theorem (Properties of congruences). If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$\begin{aligned} a + c &\equiv b + d \pmod{m} \\ a - c &\equiv b - d \pmod{m} \\ a \times c &\equiv b \times d \pmod{m} \\ a^k &\equiv b^k \pmod{m} \end{aligned}$$

This theorem is the basis for *modular arithmetic*. We do the regular operations of arithmetic (except division, which is a little trickier), but instead of using all the numbers, we reduce our operations to m equivalence classes. For example,

$$45 + 20 \equiv 3 + 5 \equiv 8 \equiv 1 \pmod{7}$$

All very pretty, but how is it of any use? One use, before the advent of the pocket calculator, was to check long calculations for errors using *digital sums*. Let's see how this works.

The digital sum of a number is obtained by adding all the digits of the number. If the result is larger than 9, the process is repeated until the result is between 1 and 9 inclusive. For example, to calculate the digital sum of 43 658 912, we would first calculate $4+3+6+5+8+9+1+2 = 38$; then, since the result is larger than 9, we would calculate $3 + 8 = 11$; and finally, $1 + 1 = 2$, which is the digital sum. You may be surprised to find out that if you calculate the remainder when 43 658 912 is divided by 9, the result is also 2.

To check a calculation, you can find the digital sums of the numbers involved. For example, if a friend of yours has calculated

$$23\,495 \times 103\,621 = 2\,433\,505\,395\,,$$

you would look at the digital sums of the two numbers being multiplied, as well as the digital sum of the answer. You would get 5, 4, and 3, respectively. But, $5 \times 4 = 20$, which has a digital sum of 2. Since that does not match the digital sum of the answer, the answer must be wrong.

This method does not always work. If the digital sums match, there may still be an error (try to come up with an example). On the other hand, if the digital sums do not match, you are certain that the answer is wrong.

For homework, try to determine why the method of the digital sums works. (*Hint:* it is related to doing arithmetic modulo 9). Next time we will look at how we can use modular arithmetic to develop divisibility rules.

Iterating Möbius Functions with Rational Coefficients, Part II

Kun-Chieh Wang

A Möbius function with rational coefficients is a function of the form $f(z) = \dfrac{az+b}{cz+d}$, where z is a complex variable and a, b, c, and d are rational numbers such that $ad \neq bc$. For such a function f, we consider the sequence $\{f_k\}_{k=0}^{\infty}$ of functions, where f_0 is the identity function, $f_1 = f$, and $f_k = f \circ f_{k-1}$ for $k \geq 2$. The sequence $\{f_k\}$ is said to be periodic if there exists a positive integer n such that $f_n = f_0$. The smallest such integer n is the period of the sequence.

In Part I, we found Möbius functions with rational coefficients that generate sequences with periods 1, 2, 3, 4, and 6, and we proved that periods 5, 8, and 12 are not possible. We then made a conjecture, which we now prove.

Theorem. Every periodic sequence $\{f_k\}$ generated by iterating a Möbius function f with rational coefficients has period 1, 2, 3, 4, or 6.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant of A is $\det A = ad - bc$; the *trace* of A is $\operatorname{tr} A = a + d$. The *characteristic equation* of A is $\det(A - xI) = 0$, where I is the 2×2 identity matrix. The roots of the characteristic equation are the *eigenvalues* of A.

We will need to use some well-known results from linear algebra, which we state without proof. For any two 2×2 matrices A and B, we have $\det(AB) = \det(A)\det(B)$. For any 2×2 matrix A, the eigenvalues are λ and μ if and only if $\lambda + \mu = \operatorname{tr}(A)$ and $\lambda\mu = \det(A)$. If λ and μ are the eigenvalues of a 2×2 matrix A, then λ^k and μ^k are the eigenvalues of A^k, for any positive integer k.

We will also need the following results.

Lemma 1. If there is a sequence $\{f_k\}$ of period n obtained by iterating a Möbius function f with rational coefficients, then there is such a sequence of period m for every positive divisor m of n.

Proof: Let $n = m\ell$. Suppose $A^n = \alpha I$, where A is the coefficient matrix of f and α is some non-zero rational number. Let $B = A^\ell$. Then $B^m = \alpha I$. If $B^j = \beta I$ for some $j < m$ and some rational $\beta \neq 0$, then $A^{j\ell} = \beta I$. This is not possible, since $j\ell < n$ and n is the period of $\{f_k\}$. ∎

Lemma 2. Let n be an odd positive integer. If there is a Möbius function f with rational coefficients such that iteration of f generates a sequence $\{f_k\}$ of period n, then there is such a function f with a coefficient matrix B such that $B^n = I$.

Proof: Let f be a Möbius function with rational coefficients such that iteration of f generates a sequence $\{f_k\}$ of period n, and let A be the coefficient matrix of f. Then $A^n = \alpha I$ for some rational number $\alpha \neq 0$. Note that $A^{2n} = \alpha^2 I$ and $\alpha^2 = \det(\alpha I) = \det(A^n) = (\det A)^n$. Hence, $\alpha^{2/n} = \det A$, which is a non-zero rational number. Let g be the Möbius function whose coefficient matrix is $B = \frac{1}{\alpha^{2/n}} A^2$, and let $\{g_k\}$ be the sequence obtained by iterating g. The entries of B are rational and $B^n = I$.

Now suppose that $B^k = \beta I$ for some positive divisor k of n and some rational number $\beta \neq 0$. Then $A^{2k} = \left(\alpha^{2/n}\right)^k \beta I$. Hence, $2k \geq n$. Since n is odd, $k > n/2$. Then $k = n$. We conclude that $\{g_k\}$ has period n. ∎

An important tool is the following trigonometric result.

Lemma 3. If both θ/π and $\cos\theta$ are rational, then $\cos\theta \in \{0, \pm 1, \pm\frac{1}{2}\}$.

Proof: Let $\theta/\pi = m/n$ where m is an integer and n is a positive integer. Then $\cos n\theta = \cos m\pi = (-1)^m$. Trigonometric identities yield

$$2\cos n\theta = (2\cos\theta)^n + a_1(2\cos\theta)^{n-1} + \cdots + a_{n-1}(2\cos\theta) + a_n\,,$$

where the coefficients a_i are integers. Now $2\cos\theta$ is a rational root of the monic polynomial $x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ with integer coefficients. Hence, $2\cos\theta$ must be an integer. It follows that $\cos\theta \in \{0, \pm 1, \pm\frac{1}{2}\}$.

Proof of Theorem: Let A be the coefficient matrix of f. Suppose $A^n = \alpha I$ for some odd positive integer n and some non-zero rational number α. By Lemma 2, we may take $\alpha = 1$. Then $(\det A)^n = \det(A^n) = \det I = 1$. Since n is odd, $\det A = 1$. Let λ and μ be the eigenvalues of A. Then λ^n and μ^n are the eigenvalues of $A^n = I$. Hence, $\lambda^n = 1$ and $\mu^n = 1$. Now $\lambda\mu = \det(A) = 1$. Since both λ and μ are n^{th} roots of unity, $\mu = \overline{\lambda}$. We also have $\lambda + \mu = \operatorname{tr} A$, which is rational.

Let $\lambda = \cos\theta + i\sin\theta$, where $\theta = 2t\pi/n$ for some integer t such that $0 \leq t < n$. Note that we have $\cos\theta = \frac{1}{2}\left(\lambda + \overline{\lambda}\right) = \frac{1}{2}\left(\lambda + \mu\right) = \frac{1}{2}\operatorname{tr} A$. This is rational, as is $\theta/\pi = 2t/n$. By Lemma 3, $\cos(2t\pi/n) \in \{0, \pm 1, \pm\frac{1}{2}\}$. Hence, the only possible odd values for n are 1 or 3.

By Lemma 1 and the result on odd values for n, the only possible even values for n must have the form 2^u or $2^u 3$ with $u \geq 1$. By Lemma 1 again, the elimination of 8 and 12 eliminates all but 2, 4, and 6 as possible even values for n. This completes the proof of the theorem. ∎

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THE OLYMPIAD CORNER

No. 245

R.E. Woodrow

This number starts with the problems of the XX Colombian Mathematical Olympiad, Higher Level, June 7–8, 2001. My thanks go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting them for our use.

XX COLOMBIAN MATHEMATICAL OLYMPIAD
Higher Level
June 7–8, 2001

1. [7 points] Let ABC be an isosceles triangle with $AB = AC$. Let M be the mid-point of side BC. The circle with diameter AB cuts side AC at point P. The parallelogram $MPDC$ is constructed so that $PD = MC$ and $PD \parallel MC$. Prove that triangles APD and APM are congruent.

2. [7 points] Find all positive integers z for which the equation

$$x(x+z) \;=\; y^2$$

has no solutions x, y that are positive integers.

3. [7 points] Let $n \geq 4$ be a fixed integer. Let $S = \{P_1, P_2, \ldots, P_n\}$ be a set of n points in the plane, no three of which are collinear and no four concyclic. Let a_t, $1 \leq t \leq n$, be the number of circles $P_iP_jP_k$ that contain P_t in the interior, and let

$$m(S) \;=\; a_1 + a_2 + \cdots + a_n \,.$$

Prove that there exists a positive integer $f(n)$, depending only on n, such that the points of S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

4. [7 points] Let x and y be any two real numbers. Prove that

$$3(x+y+1)^2 + 1 \;\geq\; 3xy \,.$$

Under what conditions does equality hold?

5. [7 points] Let b be an odd positive integer, and let $a = \dfrac{b^2-1}{4}$. For each positive integer $n > \sqrt{a}$, define the sequence n_0, n_1, $\ldots$, n_k, $\ldots$ in the following way: $n_0 = n$, and $n_i = n_{i-1}^2 - a$ for $i \geq 1$.

Determine all values of n for which there exists a positive integer k such that $n_k = n$.

6. Mr. Leonardo invited a group of children to go for a ride around a lake on his boat, in several turns. He later realized that the following things had happened:

- In each turn, there had been exactly three children on the boat.
- Each pair of children had been together on the boat in exactly one turn.

(a) [2 points] Prove that if Mr. Leonardo invited n children, then n must be a number of the form $6t+1$ or $6t+3$, where t is a non-negative integer.

(b) [5 points] Prove that, for any non-negative integer t, Mr. Leonardo can invite $6t+3$ children under the above conditions.

As a second problem set we give the 53$^{\text{th}}$ Polish Mathematical Olympiad 2001–02, Final Round, April 2002. Thanks go to Bill Sands, Chair of the IMO Committee of the Canadian Mathematical Society, for forwarding the collection for our use.

53$^{\text{th}}$ POLISH MATHEMATICAL OLYMPIAD 2001-02

Final Round

April 3-4, 2002

1. Determine all triples of positive integers a, b, c such that a^2+1 and b^2+1 are prime numbers satisfying $(a^2+1)(b^2+1) = c^2+1$.

2. On sides AC and BC of an acute-angled triangle ABC, rectangles $ACPQ$ and $BKCL$ are erected outwardly. Assuming that these rectangles have equal areas, show that the vertex C, the circumcentre of triangle ABC, and the mid-point of segment PL are collinear.

3. Three non-negative integers are written on a board. Two of them, k and m, are chosen. These two are erased and replaced by $k+m$ and $|k-m|$, while the third number remains unchanged. The same procedure is applied to the resulting triple of numbers, and so on. Determine whether it is always possible, given any initial triple of non-negative integers, to obtain a triple with at least two zeros.

4. Prove that, for every integer $n \geq 3$ and every sequence of positive numbers x_1, x_2, $\ldots$, x_n, at least one of the following two inequalities is satisfied:

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \geq \frac{n}{2}, \qquad \sum_{i=1}^{n} \frac{x_i}{x_{i-1}+x_{i-2}} \geq \frac{n}{2}.$$

(Note: Here $x_{n+1} = x_1$, $x_{n+2} = x_2$, $x_0 = x_n$, and $x_{-1} = x_{n-1}$.)

5. In space we are given a triangle ABC and a sphere s disjoint from the plane ABC. Let K, L, M, and P be points on s such that AK, BL, CM are tangent to s, and

$$\frac{AK}{AP} = \frac{BL}{BP} = \frac{CM}{CP}\,.$$

Prove that the circumsphere of the tetrahedron $ABCP$ is tangent to s.

6. Let k be a fixed positive integer. The infinite sequence $\{a_n\}$ is defined by the formulae $a_1 = k + 1$ and $a_{n+1} = a_n^2 - ka_n + k$ for $n \geq 1$. Show that if $m \neq n$, then the numbers a_m and a_n are relatively prime.

As a third set for your puzzling pleasure, we give the 2nd Czeck-Polish-Slovak Mathematical Competition, June 17–18, 2002. Thanks again go to Bill Sands for obtaining these problems.

2nd CZECH-POLISH-SLOVAK MATHEMATICAL COMPETITION
Zwardoń, Poland
June 17–18, 2002

1. Let a and b be distinct real numbers, and let k and m be positive integers with $k + m = n \geq 3$, $k \leq 2m$, $m \leq 2k$. We consider sequences $x_1, \dots, x_n$ with the following properties:

- k terms x_i are equal to a; in particular, $x_1 = a$;
- m terms x_i are equal to b; in particular, $x_n = b$;
- no three consecutive terms are equal.

Determine all possible values of the sum

$$x_n x_1 x_2 + x_1 x_2 x_3 + \cdots + x_{n-1} x_n x_1\,.$$

2. A given triangle ABC has area S and sidelengths $BC = a$, $CA = b$, and $AB = c$, where $a \leq b \leq c$. Determine the greatest number u and the least number v such that, for every point P inside triangle ABC, the inequality $u \leq PD + PE + PF \leq v$ holds, where D, E, F are the (respective) points of intersection of the rays AP, BP, CP with the opposite sides of the triangle. (The required values of u and v have to be expressed in terms of the given data a, b, c, S.)

3. Let n be a given positive integer, and let $S = \{1, 2, \dots, n\}$. How many functions $f : S \to S$ are there such that $x + f^4(x) = n + 1$ for all $x \in S$? Note: The symbol f^4 denotes the fourth iterate: $f^4(x) = f\big(f\big(f(f(x))\big)\big)$.

4. An integer $n > 1$ and a prime p are such that n divides $p - 1$, and p divides $n^3 - 1$. Show that $4p - 3$ is the square of an integer.

5. In an acute-angled triangle ABC with circumcentre O, points P and Q lying respectively on sides AC and BC are such that

$$\frac{AP}{PQ} = \frac{BC}{AB} \quad \text{and} \quad \frac{BQ}{PQ} = \frac{AC}{AB}.$$

Show that the points O, P, Q, and C are concyclic.

6. Let $n \geq 2$ be a fixed even integer. We consider polynomials of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$$

with real coefficients, having at least one real root. Determine the least possible value of the sum $a_1^2 + \cdots + a_{n-1}^2$.

Next we have a correction.

23. [2002 : 201–203; 2004 : 360] *St. Petersburg Contests 1965–1984*. The plane is divided into regions by n lines in general positions. Prove that at least $n - 2$ of the regions are triangles.

Correction by Pierre Bornsztein, Maisons-Laffitte, France.

My solution published in [2004 : 360] is totally erroneous. The 'non-destroying triangle' argument does not work at all. An excellent survey of this problem, its solution, and other erroneous attempts appeared in *Quantum*, March/April 2001, pp. 10–18.

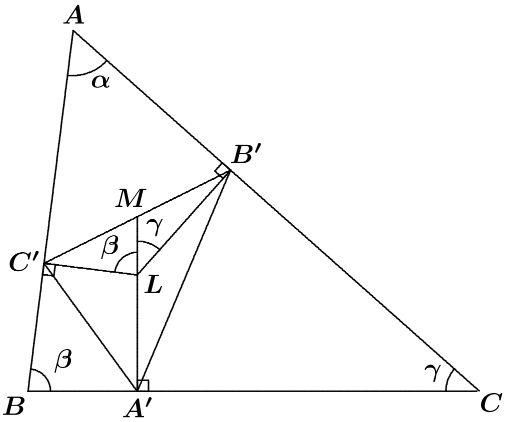
Now we look at solutions from the readers to problems of the 2000 Bulgarian Mathematical Olympiad given [2003 : 23–24].

2. Let ABC be an acute triangle.

(a) Prove that there exist unique points A', B', and C', on BC, CA, and AB, respectively, such that A' is the mid-point of the orthogonal projection of $B'C'$ onto BC, B' is the mid-point of the orthogonal projection of $C'A'$ onto CA, and C' is the mid-point of the orthogonal projection of $A'B'$ onto AB.

(b) Prove that $A'B'C'$ is similar to the triangle formed by the medians of ABC.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.



The points A', B', C' are the projections onto BC, CA, AB of the Lemoine Point (or symmedian point) L of $\triangle ABC$. We denote

$$\lambda = \frac{LA'}{a} = \frac{LB'}{b} = \frac{LC'}{c}.$$

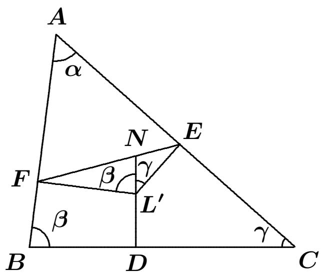
(a) Let M be the point of intersection of $A'L$ and $B'C'$. Rectangle $BA'LC'$ is cyclic, which implies that $\angle C'LM = \beta$. Rectangle $CA'LB'$ is cyclic, implying that $\angle B'LM = \gamma$.

By the Sine Law, first in $\triangle B'C'L$ and then in $\triangle ABC$, we have

$$\frac{\sin\angle LC'B'}{\sin\angle LB'C'} = \frac{LB'}{LC'} = \frac{\lambda b}{\lambda c} = \frac{\sin\beta}{\sin\gamma} = \frac{\sin\angle C'LM}{\sin\angle B'LM}. \qquad (1)$$

Equation (1) is a necessary and sufficient condition for LM to be a median in $\triangle B'C'L$. It follows that A' is the mid-point of the orthogonal projection of $B'C'$ onto BC. By symmetry, analogous statements are true for B' and C'.

Now we will prove the uniqueness. Let L' be a point not on AL, and denote its projections onto BC, CA, and AB by D, E, and F respectively. Let DL' intersect EF at N.



Since $L'E : L'F \neq b : c$, equation (1) does not hold for $\triangle L'EF$, and N is not the mid-point of EF. Thus, A', B', C' are unique indeed.

(b) Let us denote the length of the median from A to BC by m_a. Applying the Law of Cosines, first in $\triangle B'C'L$ and then in $\triangle ABC$, we obtain

$$\begin{aligned}(B'C')^2 &= (LB')^2 + (LC')^2 - 2(LB')(LC')\cos(\beta+\gamma)\\ &= \lambda^2(b^2 + c^2 + 2bc\cos\alpha)\\ &= 4\lambda^2\left(\tfrac{1}{2}(b^2+c^2) - \tfrac{1}{4}a^2\right)\\ &= 4\lambda^2 m_a^2\,;\end{aligned}$$

whence, $B'C' = \lambda m_a$. Similarly, $C'A' = \lambda m_b$ and $A'B' = \lambda m_c$, where m_b and m_c are the lengths of the medians (in $\triangle ABC$) to sides b and c, respectively. Therefore, $\triangle A'B'C'$ is similar to the triangle formed by the medians of $\triangle ABC$.

4. Find all polynomials $P(x)$ with real coefficients such that we have $P(x)P(x+1) = P(x^2)$ for all real x.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

If $P \equiv 0$ or $P \equiv 1$, then P is clearly a solution. These are the only constant polynomials which are solutions of the problem.

Now suppose that P is a non-constant polynomial which is a solution. Then $P(x)P(x+1) = P(x^2)$ for all real x. Hence, $P(z)P(z+1) = P(z^2)$ for all $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be a root of P. Then, $P(z^2) = 0$ and $P\big((z-1)^2\big) = 0$.

Suppose that $0 < |z| < 1$. Define a sequence $\{z_n\}_{n=0}^{\infty}$ by $z_0 = z$ and $z_{n+1} = z_n^2$ for $n \geq 0$. Then, for each $n \geq 0$, we have $0 < |z_{n+1}| < |z_n| < 1$ and $P(z_n) = 0$. Thus, P has an infinite number of distinct roots, and then $P \equiv 0$, a contradiction.

Similarly, if $|z| > 1$, then $|z_{n+1}| > |z_n| > 1$ and $P(z_n) = 0$, which leads to the same contradiction.

Now suppose that $|z| = 1$ and $z \notin \{1, e^{i\frac{\pi}{3}}, e^{-i\frac{\pi}{3}}\}$. Let $z = e^{i\theta}$. Then $|(z-1)^2| = 2(1-\cos\theta) \in (0,1) \cup (1,4)$. Considering the sequence defined by $z_0 = (1-z)^2$ and $z_{n+1} = z_n^2$ for $n \geq 0$, we use an argument like the one above to get a contradiction.

Thus, the only possible roots of P are 0, 1, $e^{i\frac{\pi}{3}}$, and $e^{-i\frac{\pi}{3}}$. Since P has real coefficients, we have $P(e^{i\frac{\pi}{3}}) = 0$ if and only if $P(e^{-i\frac{\pi}{3}}) = 0$. If $e^{i\frac{\pi}{3}}$ is a root, then $P(e^{i\frac{\pi}{6}})P(e^{i\frac{\pi}{6}}+1) = P(e^{i\frac{\pi}{3}}) = 0$, and $e^{i\frac{\pi}{6}}$ or $e^{i\frac{\pi}{6}}+1$ is a root of P. Since these numbers do not belong to $\{0, 1, e^{i\frac{\pi}{3}}, e^{-i\frac{\pi}{3}}\}$, we have a contradiction.

Now the only possible roots are 0 and 1. Thus, $P(x) = ax^p(x-1)^q$, where a is a non-zero real number, and p and q are non-negative integers. Then $P(x)P(x+1) = P(x^2)$ is equivalent to

$$a^2x^{p+q}(x-1)^p(x+1)^q \;=\; ax^{2p}(x-1)^q(x+1)^q\,.$$

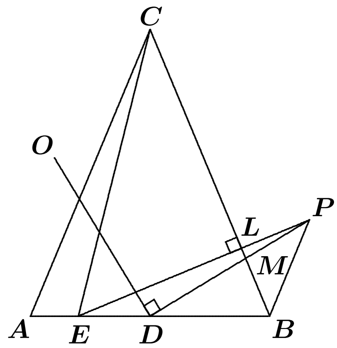
It follows that $a = 1$ and $p = q$. Thus, $P(x) = x^p(x-1)^p$. The integer p must now be positive, since P is supposed to be non-constant.

In conclusion, the solutions are $P(x) = 0$ and $P(x) = x^p(x-1)^p$, where p may be any non-negative integer.

5. In triangle ABC, we have $CA = CB$. Let D be the mid-point of AB and E an arbitrary point on AB. Let O be the circumcentre of $\triangle ACE$. Prove that the line through D perpendicular to DO, the line through E perpendicular to BC, and the line through B parallel to AC are concurrent.

Solution by Christopher J. Bradley, Bristol, UK.

Take rectangular Cartesian coordinates with $D(0,0)$, $A(-k,0)$, $B(k,0)$, $C(0,h)$, $E(t,0)$. The mid-point of AE is $\left(\frac{1}{2}(t-k),0\right)$; hence, the x–coordinate of O is $\frac{1}{2}(t-k)$.



The equation of AC is $ky - hx = hk$, and the mid-point of AC is $\left(-\frac{1}{2}k, \frac{1}{2}h\right)$. The perpendicular bisector of AC is then $hy + kx = \frac{1}{2}(h^2 - k^2)$. It follows that the coordinates of O are $\left(\frac{1}{2}(t-k), \frac{h^2-tk}{2h}\right)$. Therefore, the slope of OD is $\frac{h^2-tk}{h(t-k)}$.

The equation of the line through D perpendicular to OD is $(h^2 - tk)y = h(k-t)x$. The line through B parallel to AC has equation $ky = h(x-k)$, and the line through E perpendicular to BC has equation $hy = k(x-t)$. It is easy to check that all three lines pass through the point P with coordinates

$$\left(\frac{k(h^2-kt)}{h^2-k^2}, \frac{hk(k-t)}{h^2-k^2}\right).$$

It appears that we must exclude the case when $h = \pm k$; that is, when $\angle ACB = 90^\circ$. However, in this case, the three lines of the problem are all perpendicular to BC and thus "meet at infinity".

6. Let $\mathcal{A}$ be the set of all binary sequences of length n, and let $0 \in \mathcal{A}$ be the sequence all terms of which are zeroes. The sequence $c = \langle c_1, c_2, \ldots, c_n\rangle$ is called the sum of $a = \langle a_1, a_2, \ldots, a_n\rangle$ and $b = \langle b_1, b_2, \ldots, b_n\rangle$ if $c_i = 0$ when $a_i = b_i$ and $c_i = 1$ when $a_i \neq b_i$. Let $f : \mathcal{A} \to \mathcal{A}$ be a function such that $f(0) = 0$ and if the sequences a and b differ in exactly k terms then the sequences $f(a)$ and $f(b)$ differ also exactly in k terms. Prove that if a, b, and c are sequences from $\mathcal{A}$ such that $a+b+c = 0$, then $f(a)+f(b)+f(c) = 0$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France.

The set $\mathcal{A}$ is a vector space on $\mathbb{Z}/2\mathbb{Z}$ (the sum is the one defined in the statement, and let $0 \cdot a = 0$ and $1 \cdot a = a$) with dimension n and canonical basis $B_1 = (e_1, \ldots, e_n)$, where $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$, $\ldots$, $e_n = (0,\ldots,0,1)$. If $a = \langle a_1, a_2, \ldots, a_n\rangle$, then $a = \sum_{i=1}^{n} a_i e_i$.

Let $d(a,b) = \sum_{i=1}^{n} |a_i - b_i|$. Then $d(a,b)$ is simply the number of terms that differ in the sequences a and b. From the statement of the problem, $d\big(f(a), f(b)\big) = d(a,b)$. It follows trivially that f is injective.

Let $i \in \{1, \ldots, n\}$. Since $f(0) = 0$ and $d(0, e_i) = 1$, we deduce that $d\big(0, f(e_i)\big) = 1$. It follows that there exists $m \in \{1, \ldots, n\}$ such that $f(e_i) = e_m$. Let $f_i = f(e_i)$. Since f is injective, we deduce that

$B_2 = (f_1, \ldots, f_n)$ is a permutation of $(e_1, \ldots, e_n)$ and, therefore, is also a basis for $\mathcal{A}$.

Let $a = (a_1, a_2, \ldots, a_n)_{B_1}$ with $d(0, a) = k$. Then $d\big(0, f(a)\big) = k$. Let $i \in \{1, \ldots, n\}$.

- If $a_i = 1$, then $d(e_i, a) = k - 1$. Thus, $d\big(f_i, f(a)\big) = k - 1$; that is, the i^{th} coordinate of $f(a)$ in B_2 is equal to 1.
- If $a_i = 0$, then $d(e_i, a) = k$. Thus, $d\big(f_i, f(a)\big) = k$; that is, the i^{th} coordinate of $f(a)$ in B_2 is equal to 0.

It follows that, if $a = \sum\limits_{i=1}^{n} a_i e_i$, then $f(a) = \sum\limits_{i=1}^{n} a_i f_i = \sum\limits_{i=1}^{n} a_i f(e_i)$. Thus, f is linear.

Then, if $a + b + c = 0$, we have

$$f(a) + f(b) + f(c) \;=\; f(a + b + c) \;=\; f(0) \;=\; 0\,.$$

Next we turn to solutions from our readers to problems of the 2000 Belarusian Mathematical Olympiad given [2003 : 87–88].

1. Pete and Bill play the following game. At the beginning, Pete chooses a number a, then Bill chooses a number b, and then Pete chooses a number c. Can Pete choose his numbers in such a way that the three equations $x^3 + ax^2 + bx + c = 0$, $x^3 + bx^2 + cx + a = 0$, and $x^3 + cx^2 + ax + b = 0$ have a common

(a) real root?

(b) negative root?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Pavlos Maragoudakis, Lefkogia, Crete, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's solution.

(a) Yes. Pete simply chooses $a = -1$, and then, for any b chosen by Bill, Pete chooses $c = -b$. The three equations become $x^3 - x^2 + bx - b = 0$, $x^3 + bx^2 - bx - 1 = 0$, and $x^3 - bx^2 - x + b = 0$, which clearly have $x = 1$ as a common root.

(b) No. Suppose Bill chooses $b = 0$, and suppose r is a negative root common to all three equations. Then, in particular, we have

$$r^3 + cr + a \;=\; 0 \qquad (1)$$
$$\text{and} \quad r^3 + cr^2 + ar \;=\; 0\,. \qquad (2)$$

From (2) we get $r^2 + cr + a = 0$, or $cr + a = -r^2$. Substituting into (1), we then get $r^3 - r^2 = 0$, which yields $r = 0$ or 1, a contradiction.

2. How many pairs (n, q) satisfy $\{q^2\} = \left\{\frac{n!}{2000}\right\}$, where n is a positive integer and q is a non-integer rational number such that $0 < q < 2000$? [*Editor's comment:* $\{r\}$ means the "fractional part" of r.]

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Lefkogia, Crete, Greece. We feature Bornsztein's solution, modified by the editor.

There are 2400 such pairs.

Let n be a positive integer and q a non-integer rational number such that $0 < q < 2000$. Let a and b be positive integers such that $q = a/b$ and $\gcd(a, b) = 1$. Then $b > 1$ (since q is not an integer). We say that (n, q) is a *good pair* if $\{q^2\} = \left\{\frac{n!}{2000}\right\}$. The problem is to find how many pairs (n, q) are good pairs.

For any real number x, let $\lfloor x \rfloor = x - \{x\}$ (the integer part of x). Then (n, q) is a good pair if and only if

$$\frac{a^2}{b^2} - \left\lfloor \frac{a^2}{b^2} \right\rfloor = \frac{n!}{2000} - \left\lfloor \frac{n!}{2000} \right\rfloor ; \qquad (1)$$

that is,

$$2000a^2 = b^2 \left(2000\lfloor q^2 \rfloor + n! - 2000 \left\lfloor \frac{n!}{2000} \right\rfloor\right) .$$

If this condition is satisfied, then, according to Gauss' Theorem, b^2 divides 2000, and it follows that $b \in \{2, 4, 5, 10, 20\}$.

Suppose that $b \neq 5$. Then $b = 2\tilde{b}$, where $\tilde{b} \in \{1, 2, 5, 10\}$, and a is odd, since $\gcd(a, b) = 1$. From (1), we have

$$a^2 = 4\tilde{b}^2 \lfloor q^2 \rfloor + \frac{n!\tilde{b}^2}{500} - 4\tilde{b}^2 \left\lfloor \frac{n!}{2000} \right\rfloor .$$

It follows that $\frac{n!\tilde{b}^2}{500}$ is an odd integer. This cannot be true if $n!$ is divisible by 8; therefore, $n \leq 3$. But then $\frac{n!\tilde{b}^2}{500}$ is not an integer, a contradiction.

Thus, we must have $b = 5$. Then a is not divisible by 5. From (1), we have

$$a^2 = 25\lfloor q^2 \rfloor + \frac{n!}{80} - 25 \left\lfloor \frac{n!}{2000} \right\rfloor .$$

It follows that $\frac{n!}{80}$ is an integer not divisible by 5; that is, $n \in \{6, 7, 8, 9\}$.

We have reduced the problem to that of finding good pairs (n, q) of the form $\left(n, \frac{a}{5}\right)$, where $n \in \{6, 7, 8, 9\}$ and a is a positive integer not divisible by 5. Moreover, note that for any positive integer a,

$$\left\{\left(\frac{25 \pm a}{5}\right)^2\right\} = \left\{25 \pm 2a + \frac{a^2}{25}\right\} = \left\{\left(\frac{a}{5}\right)^2\right\} .$$

Therefore, if $\left(n, \frac{a}{5}\right)$ is a good pair, then $\left(n, \frac{25+a}{5}\right)$ is a good pair, and so is $\left(n, \frac{25-a}{5}\right)$ if $0 < a < 25$. Hence, we will be able to determine all good pairs $\left(n, \frac{a}{5}\right)$ by finding those pairs for which $1 \le a \le 12$ (and a is not divisible by 5).

Case 1. $n = 6$.

Then $\left\{\frac{n!}{2000}\right\} = \frac{9}{25}$, and $\left(6, \frac{a}{5}\right)$ is a good pair if and only if $\left\{\frac{a^2}{25}\right\} = \frac{9}{25}$. It is easy to check that $a = 3$ is the only solution satisfying $1 \le a \le 12$. It follows that the good pairs of the form $(6, q)$ are all pairs of the form $\left(6, \frac{3+25k}{5}\right)$ and $\left(6, \frac{22+25k}{5}\right)$, for $k = 0, 1, \ldots, 399$. There are 800 such good pairs.

Case 2. $n = 7$.

Then $\left\{\frac{n!}{2000}\right\} = \frac{13}{25}$, and $\left(7, \frac{a}{5}\right)$ is a good pair if and only if $\left\{\frac{a^2}{25}\right\} = \frac{13}{25}$. If this equation is satisfied, then

$$a^2 \;=\; 25\left\lfloor\frac{a^2}{25}\right\rfloor + 13 \;\equiv\; 3 \pmod 5 \;.$$

But a square is never congruent to $3 \pmod 5$. Thus, there is no good pair in this case.

Case 3. $n = 8$.

Then $\left\{\frac{n!}{2000}\right\} = \frac{4}{25}$, and $\left(8, \frac{a}{5}\right)$ is a good pair if and only if $\left\{\frac{a^2}{25}\right\} = \frac{4}{25}$. Here $a = 2$ is the only solution satisfying $1 \le a \le 12$. It follows that the good pairs of the form $(8, q)$ are all pairs of the form $\left(8, \frac{2+25k}{5}\right)$ and $\left(8, \frac{23+25k}{5}\right)$, for $k = 0, 1, \ldots, 399$. There are 800 such good pairs.

Case 4. $n = 9$.

Then $\left\{\frac{n!}{2000}\right\} = \frac{11}{25}$, and $\left(9, \frac{a}{5}\right)$ is a good pair if and only if $\left\{\frac{a^2}{25}\right\} = \frac{11}{25}$. Now $a = 6$ is the only solution satisfying $1 \le a \le 12$. It follows that the good pairs of the form $(9, q)$ are all pairs of the form $\left(9, \frac{6+25k}{5}\right)$ and $\left(9, \frac{19+25k}{5}\right)$, for $k = 0, 1, \ldots, 399$. There are 800 such good pairs, and we are done.

3. Given a fixed integer $N \ge 5$, and any sequence $e_1, e_2, \ldots, e_N$, where $e_i \in \{1, -1\}$ for $i = 1, 2, \ldots, N$, a move is made by choosing any five consecutive terms and changing their signs. Two such sequences are said to be similar if one of them can be obtained from the other in a finite number of moves. Find the maximal number of sequences no two of which are similar to each other.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Lefkogia, Crete, Greece. We give Bornsztein's solution.

The maximal number of sequences, no two of which are similar, is 16.

For $i \in \{1, 2, \ldots, N-4\}$, we define move number i to be the move that changes the signs of $e_i, \ldots, e_{i+4}$. A sequence of such moves will be represented by a sequence of move numbers, denoted by $U = (U_1, \ldots, U_p)$. The length of the sequence is $\ell(U) = p$. We also use the notation $\emptyset$ for the null sequence, which consists of no moves at all and has length $\ell(\emptyset) = 0$.

Consider an initial state $(e_1, \ldots, e_N)$ which is changed to a final state $(e'_1, \ldots, e'_N)$ by a finite sequence of moves $U = (U_1, \ldots, U_p)$. For each $i = 1, \ldots, N$, the value of e'_i differs from the value of e_i if and only if the parity of the total number of occurrences of $i-4, i-3, \ldots, i$ in the sequence U is odd. It follows that:

(a) the order of the moves has no importance;

(b) if the same move is used twice, then both applications of this move may be omitted without affecting the final state;

(c) we may suppose that U is ordered so that $U_1 < \cdots < U_p$.

We claim that whenever a sequence of moves changes an initial state $(e_1, \ldots, e_N)$ to a final state $(e'_1, \ldots, e'_N)$, there is a unique ordered sequence (as in (c) above) that produces this change.

To prove the claim, suppose that two different ordered sequences U and V produce the same change from some initial state $(e_1, \ldots, e_N)$ to a final state $(e'_1, \ldots, e'_N)$. Since U and V are different, they cannot both be null. We assume $U \neq \emptyset$.

Let $a = U_1$. Since $U_i > a$ for $i \geq 2$, the value of e_a is changed only by the first move in the sequence represented by U, while the value of e_i for any $i < a$ is not changed by any move in the sequence. Thus, $e'_a \neq e_a$ and $e'_i = e_i$ for $i < a$. In other words, $U_1 = a$ is the least index i such that $e'_i \neq e_i$. Since $e'_a \neq e_a$, the sequence V cannot be null. Applying to V the same argument that has just been applied to U, we deduce that $V_1 = a = U_1$.

Now we impose on the sequences U and V the additional condition that $\ell(U)$ is minimal. Let $p = \ell(U)$ and $q = \ell(V)$. Let $\tilde{U} = (U_1, \ldots, U_p, U_1)$ and $\tilde{V} = (V_1, \ldots, V_q, V_1)$. Then $\tilde{U}$ and $\tilde{V}$ have the same effect on the initial state $(e_1, \ldots, e_N)$ (because U and V have the same effect on this initial state and $U_1 = V_1$). Using (a) and (b), we get $\tilde{U} = (U_2, \ldots, U_p)$ and $\tilde{V} = (V_2, \ldots, V_p)$. Since $\ell(\tilde{U}) < \ell(U)$, the minimality of $\ell(U)$ implies that $\tilde{U} = \emptyset$ and, hence, $\tilde{V} = \emptyset$. Thus, $U = V$, which contradicts our hypothesis. Our claim has now been proved.

Let E denote the set of all states $(e_1, \ldots, e_N)$. Then $|E| = 2^N$. We define an equivalence relation $\sim$ on E by stating that two states are equivalent if and only if they are similar. The maximal number of sequences, no two of which are similar, is the number of equivalence classes with respect to this equivalence relation.

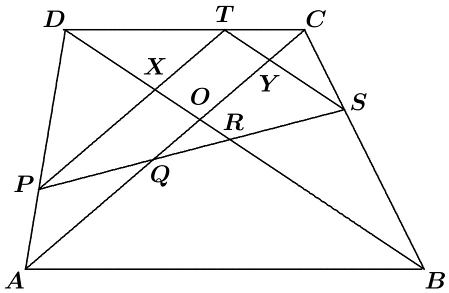
There are exactly 2^{N-4} ordered sequences of moves, including $\emptyset$ (because any such sequence either makes use of move number i or does not make use of it, for $i = 1, 2, \ldots, N-4$). By our uniqueness claim above, it follows that each equivalence class contains 2^{N-4} elements from E. Therefore, the number of classes is $\dfrac{2^N}{2^{N-4}} = 16$.

4. Let $ABCD$ be a quadrilateral with AB parallel to DC. A line ℓ intersects AD, AC, BD, and BC, forming three segments of equal lengths between consecutive points of intersection. Does it follow that ℓ is parallel to AB?

Solved by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.

We cannot conclude that l is parallel to AB. Here is a counterexample.

Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $AB = 2CD$. Let P and S be interior points of the sides AD and BC, respectively, such that $\dfrac{AP}{PD} = \dfrac{CS}{SB} \neq 1$, which implies that PS is not parallel to AB. Let Q and R be the intersections of PS with AC and BD, respectively. We shall prove that $PQ = QR = RS$.



Let T be the point on the side CD such that

$$\frac{CT}{TD} = \frac{AP}{PD} = \frac{CS}{SB}.$$

Since $\dfrac{CT}{TD} = \dfrac{AP}{PD}$, we have $PT \parallel AC$. Since $\dfrac{CT}{TD} = \dfrac{CS}{SB}$, we have $TS \parallel DB$.

Let X and O be the intersections of BD with PT and AC, respectively, and let Y be the intersection of TS with AC.

Since $XR \parallel TS$, $PT \parallel AC$, and $AB \parallel CD$, we obtain

$$\frac{PR}{RS} = \frac{PX}{XT} = \frac{AO}{OC} = \frac{AB}{CD} = \frac{2}{1}; \quad \text{hence} \quad PR = 2RS. \qquad (1)$$

Since $QY \parallel PT$, $TS \parallel DB$, and $AB \parallel CD$, we have

$$\frac{PQ}{QS} = \frac{TY}{YS} = \frac{DO}{OB} = \frac{CD}{AB} = \frac{1}{2}; \quad \text{hence} \quad QS = 2PQ. \qquad (2)$$

It follows from (1) and (2) that $PQ = QR = RS$.

7. (a) Find all positive integers n such that $(a^a)^n = b^b$ has at least one solution in integers a and b, both exceeding 1.

(b) Find all positive integers a and b such that $(a^a)^5 = b^b$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

(a) All positive integers n except $n = 2$ are solutions.

If $n = 1$, just choose $a = b \geq 2$.

If $n \geq 3$, choose $a = (n-1)^{n-1}$ and $b = (n-1)^n$. Then a, $b \geq 2$, and

$$(a^a)^n = a^{an} = (n-1)^{n(n-1)^n} = \big((n-1)^n\big)^{(n-1)^n} = b^b\,.$$

Now suppose that there exist integers a, $b \geq 2$ such that

$$(a^a)^2 = b^b\,. \tag{1}$$

We cannot have $b \leq a$, because this gives $b^b \leq a^a < (a^a)^2$, since $a > 1$. We cannot have $2a \leq b$, because then $b^b \geq (2a)^{2a} = 2^{2a}(a^a)^2 > (a^a)^2$. Thus, we must have $a < b < 2a$. It follows that a does not divide b.

Let p be a prime number which divides a. Then, from (1), p divides b. Let α and β be the exponents of p in the prime decomposition of a and b, respectively. From (1), we have $2a\alpha = b\beta$. Then $\dfrac{\alpha}{\beta} = \dfrac{b}{2a} < 1$. Thus, $\alpha < \beta$. Since this is true for each prime p dividing a, it follows that a divides b, a contradiction.

We conclude that $n = 2$ is not a solution.

(b) Clearly, $(a,b) = (1,1)$ is a solution. Now suppose that $a > 1$ and b are positive integers such that

$$(a^a)^5 = b^b\,. \tag{2}$$

As in (a), where we proved that $a < b < 2a$, we now deduce that $a < b < 5a$.

Let p be a prime number which divides a. Then from (2), p divides b. Letting α and β be the exponents of p in the prime decomposition of a and b, respectively, we obtain $\alpha < \beta$, as in (a), from which we again deduce that a divides b. Then $b = ka$, where $k \in \{2,\ 3,\ 4\}$ (since $1 < a < b < 5a$). From (2), we have $a^{5a} = (ka)^{ka}$. Thus, $a^5 = (ka)^k$, which leads to $a^{5-k} = k^k$.

If $k = 2$, we must have $a^3 = 4$, which is impossible. If $k = 3$, then we need $a^2 = 27$, which is again impossible. If $k = 4$, our equation becomes $a = 4^4 = 256$, which leads to $b = 4^5 = 1024$. Conversely, we have seen in (a) that $(4^4, 4^5)$ is a solution of (2).

Thus, the solutions of (2) are $(1,1)$ and $(256, 1024)$.

Now we turn to solutions by our readers to problems of the 2000 Taiwanese Mathematical Olympiad given [2003 : 88].

1. Find all pairs (x, y) of positive integers such that $y^{x^2} = x^{y+2}$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Note that $(1,1)$ and $(2,2)$ are solutions. We prove that these are the only solutions.

Let x and y be positive integers such that

$$y^{x^2} = x^{y+2}. \tag{1}$$

If either $x = 1$ or $y = 1$, then $x = y = 1$. Now assume that $x > 1$ and $y > 1$. From (1), we see that x and y have exactly the same prime divisors. Let $x = \prod p_i^{\alpha_i}$ and $y = \prod p_i^{\beta_i}$ be the prime decompositions of x and y. Then, from (1), for each j,

$$x^2\beta_j = (y+2)\alpha_j. \tag{2}$$

Now suppose that one of the prime divisors of x and y, say p_i, is odd. Then $\gcd(p_i, y+2) = 1$. Since $p_i^{\alpha_i}$ divides x, we see that $p_i^{2\alpha_i}$ divides x^2. From (2), we deduce that $p_i^{2\alpha_i}$ divides α_j for each j. In particular, $p_i^{2\alpha_i}$ divides α_i. It follows that $9^{\alpha_i} < p_i^{2\alpha_i} \leq \alpha_i$. But an easy induction shows that $9^n > n$ for each positive integer n. We have arrived at a contradiction.

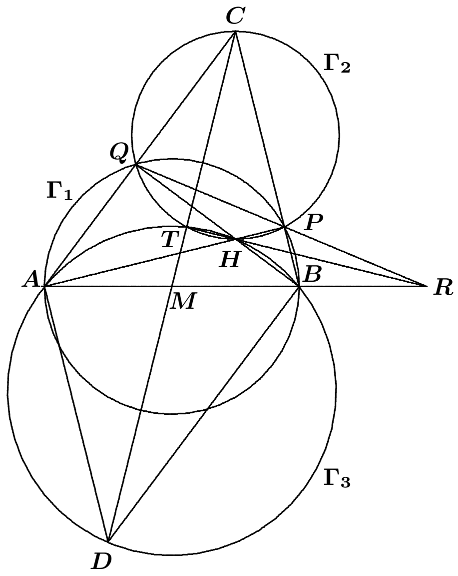
Thus, the unique prime divisor of x and y is 2. Let $x = 2^\alpha$ and $y = 2^\beta$, with α, $\beta \geq 1$. Equation (1) reduces to $2^{2\alpha-1}\beta = \alpha(2^{\beta-1}+1)$. If $\beta \geq 2$, then, since $2^{\beta-1}+1$ is odd, we see that $2^{2\alpha-1}$ divides α. But this is not possible, since $2^{2n-1} > n$ for each integer $n \geq 1$ (by an easy induction). Therefore, $\beta = 1$, and the equation reduces to $2^{2\alpha-2} = \alpha$. Since $2^{2n-2} > n$ for each integer $n \geq 2$ (by another easy induction), it follows that $\alpha = 1$. Thus, $x = y = 2$.

2. In an acute triangle ABC, $AC > BC$ and M is the mid-point of AB. Let AP be the altitude from A. Let BQ be the altitude from B meeting AP at H. Let the lines AB and PQ meet at R. Prove that the lines RH and CM are perpendicular to each other.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Lefkogia, Crete, Greece; Toshio Seimiya, Kawasaki, Japan; and Babis Stergiou, Chalkida, Greece. We first give Bataille's solution.

Since $\angle BQA = \angle BPA = 90^\circ$, the circle Γ with diameter AB passes through P and Q. Thus, the point H, as the intersection of the diagonals of $QPBA$, is on the polar of R with respect to Γ (since the sides QP and AB meet at R). Similarly, H is on the polar of C; whence, CR is the polar of H. Since M is the centre of Γ, it follows that CR is perpendicular to HM. Note that since H is the orthocentre of $\triangle ABC$, we also have $CH \perp RM$. As a result, H is the intersection of two altitudes in triangle CRM and, as such, H is the orthocentre of $\triangle CRM$. The result, $RH \perp CM$, follows.

We also give Seimiya's version.



Since $\angle APB = \angle AQB = 90^\circ$, the points A, B, P, and Q lie on a circle Γ_1. Let T be the foot of the perpendicular from H to CM. Since

$$\angle CPH = \angle CQH = \angle CTH = 90^\circ,$$

we see that C, P, Q, T, and H all lie on a circle Γ_2.

On CM produced beyond M, let D be the point such that $CM = MD$. Then quadrilateral $CADB$ is a parallelogram. Since $BD \parallel CA$ and $BQ \perp AC$, we have $BQ \perp BD$. Similarly, $AD \perp AP$. Thus,

$$\angle HBD = \angle HAD = \angle HTD = 90^\circ.$$

Therefore A, D, B, H, and T all lie on a circle Γ_3.

Note that PQ is a common chord of Γ_1 and Γ_2, HT is a common chord of Γ_2 and Γ_3, and AB is a common chord of Γ_3 and Γ_1. It follows that PQ, HT, and AB are concurrent at R. Thus, T, H, and R are collinear. Therefore, $RH \perp CM$.

3. Let $S = \{1, 2, 3, \ldots, 100\}$, and let $\mathcal{P}$ denote the family of all subsets T of S with $|T| = 49$. For each set T in $\mathcal{P}$, we label it with a number chosen at random from $\{1, 2, \ldots, 100\}$. Prove that there exists a subset M of S with $|M| = 50$ such that for each $x \in M$, $M \setminus \{x\}$ is not labelled with x.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

In general, for any positive integer n, let $S = \{1, 2, \ldots, 2n\}$, and let $\mathcal{P} = \{T \mid T \subset S, |T| = n - 1\}$. We label each set $T \in \mathcal{P}$ with a number $\ell(T)$ chosen at random from S.

Let $\mathcal{F} = \{M \mid M \subset S\,,\, |M| = n\}$. For each $T \in \mathcal{P}$, the set $T \cup \{\ell(T)\}$ is contained in $\mathcal{F}$ if and only if $\ell(T) \notin T$. The number of sets $M \in \mathcal{F}$ such that $M = T \cup \{\ell(T)\}$ for some $T \in \mathcal{P}$ is certainly no greater than the number of sets in $\mathcal{P}$, which is $\binom{2n}{n-1}$. On the other hand, the total number of sets in $\mathcal{F}$ is $\binom{2n}{n}$. Therefore, there must be at least $\binom{2n}{n} - \binom{2n}{n-1}$ sets $M \in \mathcal{F}$ such that $M \neq T \cup \{\ell(T)\}$ for any $T \in \mathcal{P}$. For any such M, if there is some $x \in M$ such that $M \setminus \{x\}$ is labelled with x, then $M = T \cup \{\ell(T)\}$ for some $T \in \mathcal{P}$, which is a contradiction.

We conclude that there are at least $\binom{2n}{n} - \binom{2n}{n-1}$ subsets M of S with $|M| = n$ such that for each $x \in M$, the set $M \setminus \{x\}$ is not labelled with x.

4. Let $\phi(k)$ denote the number of positive integers $n \le k$ such that $\gcd(n,k) = 1$. Suppose that $\phi(5^m - 1) = 5^n - 1$ for some positive integers m and n. Prove that $\gcd(m,n) > 1$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Suppose, for the purpose of contradiction, that $\gcd(m,n) = 1$. Let $d = \gcd(5^m - 1, 5^n - 1)$. Then $d = 5^{\gcd(m,n)} - 1 = 4$. Now let the prime decomposition of $5^m - 1$ be $2^a p_1^{a_1} \dots p_k^{a_k}$. Thus, $a \ge 2$ and

$$5^n - 1 \;=\; \varphi(5^m - 1) \;=\; 2^{a-1} p_1^{a_1 - 1} \cdots p_k^{a_k - 1} (p_1 - 1) \cdots (p_k - 1)\,.$$

Note that, if $5^m - 1$ is a power of 2, then $a > 3$ (otherwise, we do not have $\varphi(5^m - 1) \equiv 0 \pmod 4$), which leads to $5^n - 1 \equiv 0 \pmod 8$ and contradicts $d = 4$.

Thus, $5^m - 1$ has at least one odd prime divisor. Moreover, since $d = 4$, we must also have $a_i = 1$ for $i = 1, \dots, k$.

Case 1. m is even.

Then $5^m - 1 \equiv 0 \pmod 8$, which implies that $a \ge 3$. But, as above, we then have $2^{a-1}(p_1 - 1) \equiv 0 \pmod 8$, which forces $5^n - 1 \equiv 0 \pmod 8$ and again contradicts $d = 4$.

Case 2. $m = 2k + 1$ is odd.

Thus, $a = 2$ and, from above, $5^m - 1 = 4p_1 \cdots p_k$. Obviously, none of the p_i is equal to 5. Hence, for each i, we have $5 \times 5^{2k} = 5^m \equiv 1 \pmod{p_i}$, from which we deduce that 5 is a quadratic residue $\pmod{p_i}$. From the Quadratic Reciprocity Law, it follows that p_i is a quadratic residue $\pmod 5$.

Since, for each integer x, we have $x^2 \equiv 0, 1, \text{or} -1 \pmod 5$, we deduce that $p_i \equiv 1$ or $-1 \pmod 5$ for each i.

Now assume that there exists i such that $p_i \equiv 1 \pmod 5$. Then 5 divides $p_i - 1$, which implies that 5 divides $5^n - 1$, which is absurd. Thus, $p_i \equiv -1 \pmod 5$ for each i, and $5^m - 1 \equiv 4(-1)^k \pmod 5$; whence, k is even.

Therefore, $5^n - 1 = 2(p_1 - 1) \cdots (p_k - 1) \equiv 2(-2)^k \pmod 5$, from which we see that $5^n - 1 \equiv 2$ or $-2 \pmod 5$, a contradiction.

We conclude that $\gcd(m, n) > 1$.

Comment. Bornsztein also points out that this is problem 10626 of the American Mathematical Monthly. A solution appears in the November 1999 issue of the Monthly, p. 869.

6. Let f be a function from the set of positive integers to the set of non-negative integers such that $f(1) = 0$ and

$$f(n) \;=\; \max\{f(j) + f(n-j) + j\}$$

for all $n \geq 2$. Determine $f(2000)$.

Solved by Michel Bataille, Rouen, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's write-up.

In what follows, the maximum in the definition of $f(n)$ is considered over all j such that $1 \leq j \leq n-1$ (that is, all j for which $f(j)$ and $f(n-j)$ are defined). We will prove by induction that $f(n) = \frac{n(n-1)}{2}$ for all $n \geq 1$.

It is readily checked that $f(2) = f(1) + f(1) + 1 = 1$, that $f(3) = \max\{f(1) + f(2) + 1,\ f(2) + f(1) + 2\} = 3$, and that $f(4) = 6$. Assume that $n \geq 5$ and that $f(k) = \frac{k(k-1)}{2}$ for $1 \leq k < n$. Then

$$f(n-1) + f(1) + n - 1 \;=\; \frac{(n-2)(n-1)}{2} + 0 + n - 1 \;=\; \frac{n(n-1)}{2}\,,$$

and for $1 \leq j \leq n-2$,

$$\begin{aligned} f(j) + f(n-j) + j &= \frac{j(j-1)}{2} + \frac{(n-j)(n-j-1)}{2} + j \\ &= \frac{n(n-1) - 2j(n-1-j)}{2} \;<\; \frac{n(n-1)}{2}\,. \end{aligned}$$

It follows that $f(n) = \max\{f(j)+f(n-j)+j\} = \frac{n(n-1)}{2}$. This concludes the induction.

Now, taking $n = 2000$ in the formula we have just proved, we find that $f(2000) = 1999000$.

Remark. Wang notes that $1 + f(n+1) = 1 + \binom{n+1}{2}$, which is well known to be the number of regions into which the plane is divided by n lines in general position (every pair of lines intersect and no three lines are concurrent).

That completes the *Corner* for this issue. Send me your nice solutions and generalizations as well as Olympiad contests!

BOOK REVIEW

John Grant McLoughlin

TriMathlon: A Workout Beyond the School Curriculum
by Judith D. Sally and Paul J. Sally, Jr., published by AK Peters, 2003
ISBN 1-56881-184-5, soft cover, 200 pages, US$30.00.
Reviewed by ***Anne Izydorczak****, University at Buffalo, Amherst, NY, USA.*

In each of the ten chapters of this book, the authors introduce a mathematical situation or game. Through a series of questions and challenges followed by explanation and instruction, they gradually build toward representations, generalizations, and proofs. Students are instructed to "Pause, take time to think and to work on your own" after each question or challenge. Solutions to the questions and challenges are given in the text.

TriMathlon is divided into three areas of mathematics: arithmetic, numbers and symmetry, and geometry. In the chapter, "Lattice Polygons", students begin by attempting to make various types of polygons on a lattice. They work toward guessing and then proving Pick's Theorem. Using Pick's Theorem, they try to make shapes with specified areas. The investigation then focuses on which areas can be areas of lattice squares. This leads to a generalization and finally a proof. In the "Heavy Lifting" section at the end of the chapter, students explore equilateral triangles and prove that none can be constructed in a lattice. Some of the other explorations are palindromes, the four numbers game, circle packing in a plane, and dissection.

The book's athletic analogy is clever, but the athletic icons used throughout the book confused me at first. The section titles of run, swim, and bike do not convey much about the content of the sections. I did appreciate the personal, enthusiastic, playful tone of the book. I had the feeling that the authors were talking to me and working with me.

It is not clear for what level of student this book is intended. The authors vaguely recommend it for "young students". At first glance, the problems would seem to require some algebraic sophistication. However, they provide a meaningful context for learning algebra. The teacher can help fill in the gaps along the way. The problem contexts are likely to be motivational for students in high school or college mathematics classes, or even for prospective mathematics teachers. I would enjoy teaching from this book as a college elective.

I did find some difficulty with chapter 10, in which students do geometric constructions. In my experience, even college students have trouble with the most basic constructions. It might be better to adapt this chapter to use with a geometric software program.

This book offers a fresh approach to learning mathematics. Even the few explorations that were familiar to me went in unfamiliar directions. The book's format does more than actively involve the reader. It puts the onus for doing the mathematics on the reader. A student who works through this book will understand what it truly means to do mathematics.

The Diagonal Points of a Cyclic Quadrangle

Christopher J. Bradley

In what follows, $ABCD$ is a cyclic quadrangle in which there are no parallel sides, the circle $ABCD$ has centre O, the diagonals AC and BD meet at E, AB and CD meet at F, AD and BC meet at G, and M is the mid-point of FG. Then EFG is the diagonal-point triangle, which is self-conjugate.

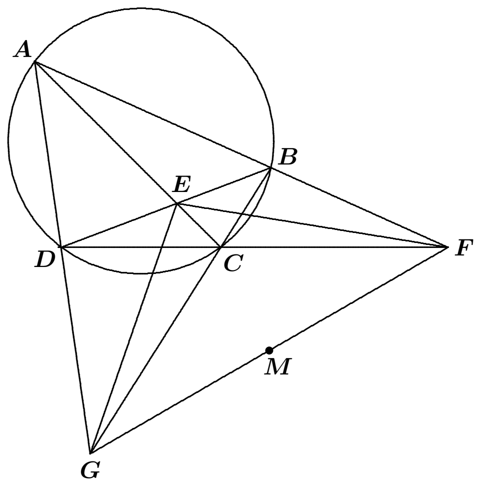


Figure 1

In this paper we are not concerned with the harmonic properties of the quadrangle, which are covered in many books, such as [1] and [2]. We do, however, set the stage for what follows by quoting, without proof, some well-known geometric properties of cyclic quadrangles. Proofs are indicated in a recent review [3].

1. *The internal angle bisectors of the angles at F and G meet at right angles.*

2. *If the tangents at A and C meet at T, and the tangents at B and D meet at U, then T, F, U, and G are collinear.*

3. *O is the orthocentre of the diagonal-point triangle.*

4. *The circles whose diameters are the sides of the diagonal-point triangle EFG are orthogonal to the circle $ABCD$.*

We refer to the set of four triangles AFG, BFG, CFG, DFG as 'Set 1' and the four triangles ACF, BDF, ACG, BDG as 'Set 2'.

Our first set of results is concerned with the triangles in Set 1.

Proposition 1. *If A', B', C', D' are the centroids of the triangles in Set 1, then $A'B'C'D'$ is homothetic with $ABCD$ and one-third the size.* (See Figure 2.)

This follows from the fact that $AA'M$, $BB'M$, $CC'M$, $DD'M$ are concurrent at M and $AA' = (2/3)AM$, etc. The circumcentres of the triangles in Set 1 are, of course, collinear on the perpendicular bisector of FG.

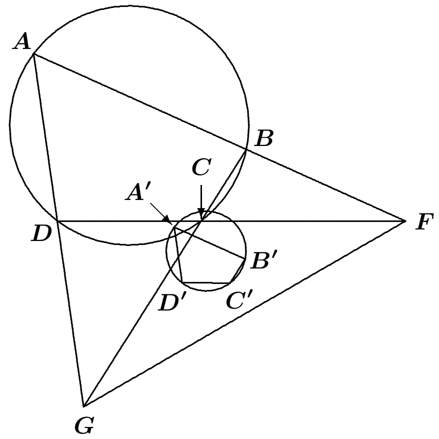


Figure 2

Proposition 2. *If A', B', C', D' are the respective orthocentres of the triangles in Set 1, then $A'B'C'D'$ is a cyclic quadrilateral with angles the same as those of $CDAB$; that is, $\angle B' = \angle D$, etc.* (See Figure 3.)

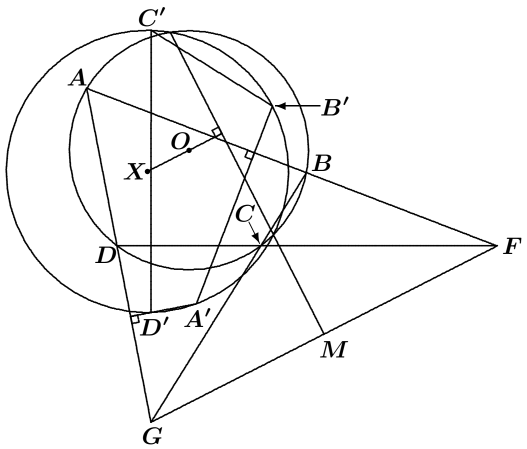


Figure 3

Since $A'B'$ is perpendicular to AB and $A'D'$ is perpendicular to AD, the angles at A and A' are supplementary, and the same is true for the other three pairs of angles. It follows that $A'B'C'D'$ is a cyclic quadrilateral. The reader is invited to show that the common chord of the two circles $ABCD$ and $A'B'C'D'$ is perpendicular to FG and passes through M. One proof uses Result 4 from the known properties of cyclic quadrangles listed above.

Conjecture. *If A', B', C', D' are the respective nine-point centres of the triangles in Set 1, then $A'D'B'C'$ is an isosceles trapezium and circle $A'D'B'C'$ passes through M.* (See Figure 4.)

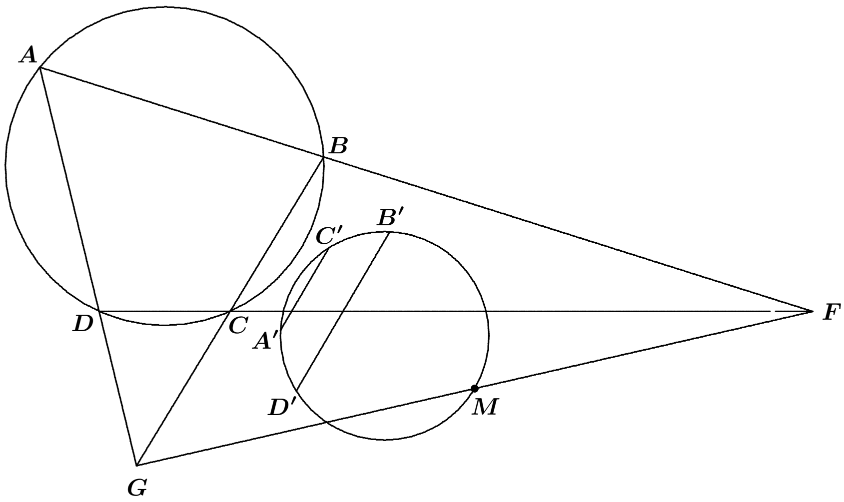


Figure 4

At the time of writing I have no proof of this conjecture. It was suggested by the software CABRI. We may be confident that the conjecture is true, such is the accuracy of the software.

Proposition 3. *If A', B', C', D' are the respective incentres of the triangles in Set 1, then $A'B'C'D'$ is a quadrilateral with opposite angles summing to an odd multiple of $\pi/2$ radians.* (See Figure 5.)

We use the simple notation $F = \angle AFG$, $G = \angle AGF$, $F' = \angle CFG$, and $G' = \angle CGF$. Then $F+G = C = 2(\pi - A')$ and $F' + G' = A = 2(\pi - C')$. Summing and dividing by 2, we obtain $(A+C) = \pi/2 = 2\pi - (A'+C')$. Therefore, $A' + C' = 3\pi/2$ and $B' + D' = \pi/2$.

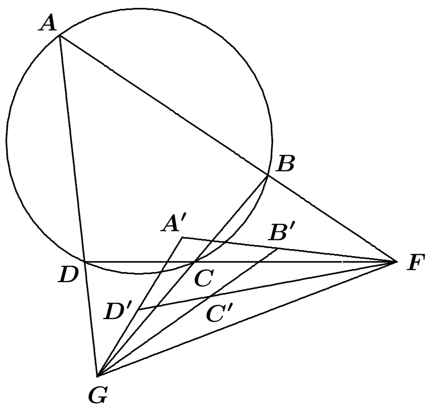


Figure 5

Our next set of results is concerned with the triangles in Set 2.

Proposition 4. *If A', B', C', D' are the respective centroids of the triangles of Set 2, then $A'B'C'D'$ is a parallelogram.*

This result may be easily proved using vectors, and it turns out that $\overrightarrow{A'C'} = \overrightarrow{B'D'} = \frac{1}{3}\overrightarrow{FG}$. Note that the result is true whether or not $ABCD$ is cyclic. The configuration consisting of the circumcentres of the triangles of Set 2 appears to have no interesting properties.

Proposition 5. *If AC and BD are perpendicular and A', B', C', D' are the respective orthocentres of the triangles of Set 2, then $A'C'B'D'$ is a parallelogram, and $A'GEB'$ and $D'EC'F$ are straight lines. Furthermore, $B'F$, AC, $D'G$ are parallel, with $B'G = D'F$; and BD, FA', $C'G$ are parallel, with $FC' = GA'$. Also, AC, FG, $A'C'$ are concurrent, as are BD, FG, $B'D'$. Finally, E is the centre of circles $A'B'F$ and $C'D'G$.* (See Figure 6.)

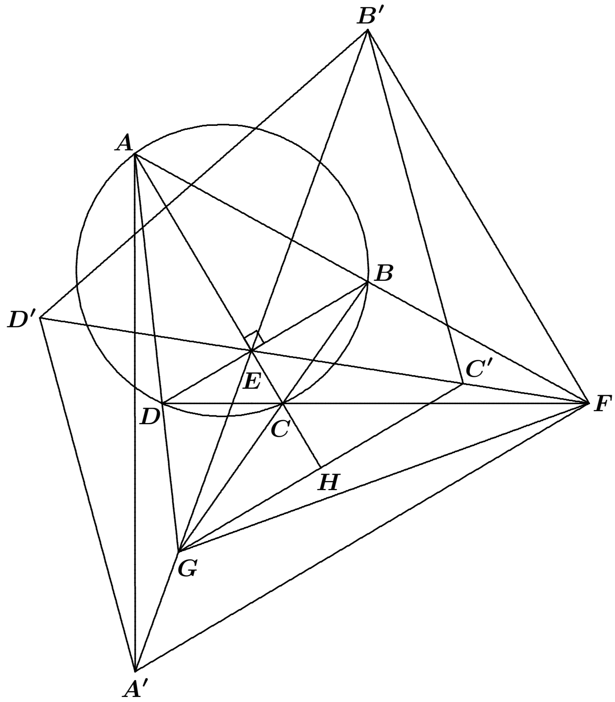


Figure 6

Two of the properties stated are easily proved. Since A' is the orthocentre of triangle ACF, it follows that FA' is perpendicular to AC and is therefore parallel to BD. Similarly, $C'G$ is parallel to BD, and both $B'F$ and $D'G$ are parallel to AC. To establish the remaining results, we use Cartesian coordinates.

Take E as the origin, and use the coordinates $A(-rs, 0)$, $B(0, pr)$, $C(pq, 0)$, and $D(0, -qs)$, where the coordinates are chosen to satisfy the intersecting chord theorem:

$$(AE)(EC) = (DE)(EB).$$

This ensures that A, B, C, D are concyclic and that AC and BD are perpendicular, with AC as the x–axis and DB as the y–axis. Details of the calculation are omitted, as it is straightforward, the difficult part being to find the coordinates of A', B', C', and D'. The points F and G turn out to have coordinates $F\big(ps(pr+qs)/(s^2-p^2), ps(pq+rs)/(s^2-p^2)\big)$ and $G\big(qr(pr+qs)/(r^2-q^2), -qr(pq+rs)/(r^2-q^2)\big)$. Note that $p \neq s$ and $r \neq q$, since AB is not parallel to CD and AD is not parallel to BC. It is worth noting that EF and EG are equally inclined to AC (a known result when AC and BD are prependicular).

After some algebra, we see that A' and B' are the reflections of F in AC and BD, respectively, and C' and D' are the reflections of G in AC and BD, respectively. Hence, there is no need to record their coordinates. From here it is easy to see that $A'B'C'D'$ is a parallelogram, that A' lies on EG, that $\angle A'FB' = \frac{\pi}{2}$, and that E is the centre of the circle $A'B'F$. All other results follow just as easily.

Proposition 6. *If AC and BD are perpendicular and A', B', C', D' are the respective nine-point centres of the triangles in Set 2, then A', D' lie on AC and C', D' lie on BD.*

Our proof of Proposition 6 relies on the following fact, whose proof is left to the reader: If XYZ is a triangle, then the nine-point centre of the triangle lies on XY if and only if $|\angle X - \angle Y| = \frac{\pi}{2}$. Using the angle properties of circle $ABCD$ and the fact that AC and BD are right angles, we have

$$\begin{aligned} \angle ACF - \angle FAC &= \angle BCA + \angle BCF - \angle FAC \\ &= \angle BCA + \angle BAD - \angle FAC \\ &= \angle BCA + \angle CAD \;=\; \angle BCA + \angle CBE \;=\; \tfrac{\pi}{2}\,. \end{aligned}$$

Acknowledgment I am grateful to the referee for suggestions to improve this paper both in presentation and content.

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 November 2005**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

3007. Correction. *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle, and let $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$ such that

$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k > 0\,.$$

1. Prove that the segments AA_1, BB_1, CC_1 are the sides of a triangle.

Let T_k denote this triangle. Let R_k and r_k be the circumradius and inradius of T_k. Prove that:

2. $P(T_k) < P(ABC)$, where $P(T)$ denotes the perimeter of triangle T;
3. $[T_k] = \dfrac{k^2+k+1}{(k+1)^2}[ABC]$, where $[T]$ denotes the area of triangle T;
4. $R_k \geq \dfrac{k\sqrt{k}\,P(ABC)}{(k+1)(k^2+k+1)}$;
5. $r_k > \dfrac{k^2+k+1}{(k+1)^2}\,r$, where r is the inradius of $\triangle ABC$.

3026. *Proposed by Michel Bataille, Rouen, France.*

Let $a > 0$. Prove that

$$\frac{a^2+1}{e^a} + \frac{3a^2-1}{3e^{3a}} + \frac{5a^2+1}{5e^{5a}} + \frac{7a^2-1}{7e^{7a}} + \cdots < \frac{\pi}{4}\,.$$

3027. *Proposed by Geoffrey A. Kandall, Hamden, CT, USA.*

Let $ABCD$ be any quadrilateral, and let M be the mid-point of AB. On the sides CB, DC, and AD, equilateral triangles CBE, DCF, and ADG are constructed externally. Let N be the mid-point of EF and P be the mid-point of FG.

Prove that $\triangle MNP$ is equilateral.

3028. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let $S_k = 1+2+\cdots+k$. Prove the following

$$1+\frac{(a_1a_2^2)^{\frac{1}{S_2}}}{a_1+2a_2}+\frac{(a_1a_2^2a_3^3)^{\frac{1}{S_3}}}{a_1+2a_2+3a_3}+\cdots+\frac{(a_1a_2^2\cdots a_n^n)^{\frac{1}{S_n}}}{a_1+2a_2+\cdots+na_n} \leq \frac{2n}{n+1}.$$

3029. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Let $a_1, a_2, \ldots, a_n$ be real numbers greater than -1, and let α be any positive real number. Prove that if $a_1+a_2+\cdots+a_n \leq \alpha n$, then

$$\frac{1}{a_1+1}+\frac{1}{a_2+1}+\cdots+\frac{1}{a_n+1} \geq \frac{n}{\alpha+1}.$$

3030. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Show that, if $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$\frac{1}{a_1}+\frac{2}{(a_2)^{\frac{1}{2}}}+\frac{3}{(a_3)^{\frac{1}{3}}}+\cdots+\frac{n}{(a_n)^{\frac{1}{n}}} \geq \frac{S_n}{(a_1a_2\cdots a_n)^{\frac{1}{S_n}}}$$

where $S_n = 1+2+\cdots+n$.

3031. *Proposed by Neven Jurič, Zagreb, Croatia.*

A quadruple (a,b,c,d) of positive integers is said to have the *Diophantine property* if each of the six integers $ab+1$, $ac+1$, $ad+1$, $bc+1$, $bd+1$, $cd+1$ is a perfect square. For example, each of the following nine quadruples has the Diophantine property:

$$\begin{array}{lll} (3,5,16,1008), & (3,8,21,2080), & (3,16,33,6440), \\ (3,21,40,10208), & (3,33,56,22360), & (3,40,65,31416), \\ (3,56,85,57408), & (3,65,96,75208), & (3,85,120,122816). \end{array}$$

Find a general expression for the sequence of quadruples (a_n,b_n,c_n,d_n) which have the Diophantine property and for which the above examples represent the first terms.

3032. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c be non-negative real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca} \leq \frac{9}{2}.$$

3033. *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Let I be the incentre of $\triangle ABC$, and let R and r be its circumradius and inradius, respectively. Prove that

$$6r \;\le\; AI+BI+CI \;\le\; \sqrt{12(R^2-Rr+r^2)}\,.$$

3034. *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Let a, b, c, x, y, z be positive real numbers. Prove that

$$\begin{aligned}&(bc+ca+ab)(yz+zx+xy)\\ &\quad\ge\; bcyz+cazx+abxy+2\sqrt{abcxyz(a+b+c)(x+y+z)}\,,\end{aligned}$$

and determine when equality occurs.

3035. *Proposed by Ali Feiz Mohammadi, student, University of Toronto, Toronto, ON.*

Are there infinitely many prime numbers that cannot be written as the sum of a prime number and a power of 2?

3036. *Proposed by Virgil Nicula, Bucharest, Romania.*

Let A, B, C be three distinct collinear fixed points. Let M be an arbitrary point not on the line ABC. The internal angle bisector of $\angle MAB$ intersects the line MB at a point X. The perpendicular at A to the line AX intersects the line MC at a point Y.

(a) Prove that the line XY passes through a fixed point D.

(b) Let Z be the projection of the point A onto the line XY. Prove that $\angle BZD = \angle CZD$.

3037. *Proposed by Ali Feiz Mohammadi, student, University of Toronto, Toronto, ON.*

There are 2005 senators in a senate. Each senator has enemies within the senate. Prove that there is a non-empty subset K of senators such that for every senator in the senate, the number of enemies of that senator in the set K is an even number.

3038. *Proposed by Virgil Nicula, Bucharest, Romania.*

Consider a triangle ABC in which $a=\max\{a,b,c\}$. Prove that the expressions $(a+b+c)\sqrt{2}-\left(\sqrt{a+b}+\sqrt{a-b}\right)\cdot\left(\sqrt{a+c}+\sqrt{a-c}\right)$ and $b^2+c^2-a^2$ have the same sign.

. . . .

3007. Correction. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit ABC un triangle et soit $A_1 \in BC$, $B_1 \in CA$ et $C_1 \in AB$ tels que

$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k > 0\,.$$

1. Montrer que les segments AA_1, BB_1 et CC_1 sont les côtés d'un triangle.

Désignons par T_k ce triangle, par r_k et R_k le rayon des cercles inscrit et circonscrit de T_k. Montrer que

2. $P(T_k) < P(ABC)$, où $P(T)$ désigne le périmètre du triangle T ;
3. $[T_k] = \dfrac{k^2+k+1}{(k+1)^2}[ABC]$, où $[T]$ désigne l'aire du triangle T ;
4. $R_k \geq \dfrac{k\sqrt{k}\,P(ABC)}{(k+1)(k^2+k+1)}$;
5. $r_k > \dfrac{k^2+k+1}{(k+1)^2}\,r$, où r désigne le rayon du cercle inscrit de $\triangle ABC$.

3026. *Proposé par Michel Bataille, Rouen, France.*

Soit $a > 0$. Montrer que

$$\frac{a^2+1}{e^a} + \frac{3a^2-1}{3e^{3a}} + \frac{5a^2+1}{5e^{5a}} + \frac{7a^2-1}{7e^{7a}} + \cdots < \frac{\pi}{4}\,.$$

3027. *Proposé par Geoffrey A. Kandall, Hamden, CT, USA.*

Soit $ABCD$ un quadrilatère quelconque, et soit M le point milieu de AB. Sur les côtés CB, DC, and AD, on construit extérieurement les triangles équilatéraux CBE, DCF et ADG. Soit N le point milieu de EF et P le point milieu de FG.

Montrer que le triangle MNP est équilatéral.

3028. *Proposé par Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Roumanie.*

Soit $S_k = 1 + 2 + \cdots + k$, et soit $a_1, a_2, \ldots, a_n$ des nombres réels positifs. Montrer que

$$1 + \frac{(a_1a_2^2)^{\frac{1}{S_2}}}{a_1+2a_2} + \frac{(a_1a_2^2a_3^3)^{\frac{1}{S_3}}}{a_1+2a_2+3a_3} + \cdots + \frac{(a_1a_2^2\cdots a_n^n)^{\frac{1}{S_n}}}{a_1+2a_2+\cdots+na_n} \leq \frac{2n}{n+1}\,.$$

3029. *Proposé par Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Roumanie.*

Soit $a_1, a_2, \ldots, a_n$ des nombres réels plus grands que -1, et soit α un nombre réel positif quelconque. Si $a_1 + a_2 + \cdots + a_n \leq \alpha n$, montrer que

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1} \;\geq\; \frac{n}{\alpha+1}\,.$$

3030. *Proposé par Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Roumanie.*

Montrer que, si $a_1, a_2, \ldots, a_n$ sont des nombres réels positifs, alors

$$\frac{1}{a_1} + \frac{2}{(a_2)^{\frac{1}{2}}} + \frac{3}{(a_3)^{\frac{1}{3}}} + \cdots + \frac{n}{(a_n)^{\frac{1}{n}}} \;\geq\; \frac{S_n}{(a_1a_2\cdots a_n)^{\frac{1}{S_n}}}$$

où $S_n = 1 + 2 + \cdots + n$.

3031. *Proposé par Neven Jurič, Zagreb, Croatie.*

On dit qu'un quadruplet (a, b, c, d) d'entiers positifs possède la *propriété diophantienne* si chacun des six entiers $ab+1$, $ac+1$, $ad+1$, $bc+1$, $bd+1$ et $cd+1$ est un carré parfait. Par exemple, chacun des neuf quadruplets suivants possède la propriété diophantienne :

$(3, 5, 16, 1008)$, $(3, 8, 21, 2080)$, $(3, 16, 33, 6440)$,
$(3, 21, 40, 10208)$, $(3, 33, 56, 22360)$, $(3, 40, 65, 31416)$,
$(3, 56, 85, 57408)$, $(3, 65, 96, 75208)$, $(3, 85, 120, 122816)$.

Trouver une expression générale pour la suite de quadruplets (a_n, b_n, c_n, d_n) qui possèdent la propriété diophantienne et pour laquelle les exemples ci-dessus représentent les premiers termes.

3032. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit a, b et c des nombres réels non négatifs tels que $a^2 + b^2 + c^2 = 1$. Montrer que

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-ca} \;\leq\; \frac{9}{2}\,.$$

3033. *Proposé par Eckard Specht, Université Otto-von-Guericke, Magdeburg, Allemagne.*

Soit I le centre et r le rayon du cercle inscrit du triangle ABC, et soit R le rayon du cercle circonscrit. Montrer que

$$6r \;\leq\; AI + BI + CI \;\leq\; \sqrt{12(R^2 - Rr + r^2)}\,.$$

3034. *Proposé par Eckard Specht, Université Otto-von-Guericke, Magdeburg, Allemagne.*

Soit a, b, c, x, y et z des nombres réels positifs. Montrer que

$$\begin{aligned}&(bc + ca + ab)(yz + zx + xy)\\ &\quad \geq \; bcyz + cazx + abxy + 2\sqrt{abcxyz(a+b+c)(x+y+z)}\,,\end{aligned}$$

et trouver quand l'égalité a lieu.

3035. *Proposé par Ali Feiz Mohammadi, étudiant, Université de Toronto, Toronto, ON.*

Y-a-t'il un nombre infini de nombres premiers qu'on ne peut pas écrire comme une somme d'un nombre premier et d'une puissance de 2 ?

3036. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

On donne trois points alignés distincts A, B et C . Soit M un point arbitraire non situé sur la droite ABC. La bissectrice de l'angle MAB coupe la droite MB en un point X. La perpendiculaire en A à la droite AX coupe la droite MC en un point Y.

(a) Montrer que la droite XY passe par un point fixe D.

(b) Soit Z la projection du point A sur la droite XY. Montrer que les angles BZD et CZD sont égaux.

3037. *Proposé par Ali Feiz Mohammadi, étudiant, Université de Toronto, Toronto, ON.*

Il y a 2005 sénateurs dans un sénat. Chaque sénateur a des ennemis à l'intérieur du sénat. Montrer qu'il y a un sous-ensemble non vide K de sénateurs tel que, pour chaque membre du sénat, le nombre d'ennemis de ce membre qui font partie de K est un nombre pair.

3038. *Proposé par Virgil Nicula, Bucarest, Roumanie.*

Soit ABC un triangle dans lequel $a = \max\{a, b, c\}$. Montrer que les expressions $(a+b+c)\sqrt{2} - \left(\sqrt{a+b} + \sqrt{a-b}\right) \cdot \left(\sqrt{a+c} + \sqrt{a-c}\right)$ et $b^2 + c^2 - a^2$ ont le même signe.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2927★. [2004 : 172, 174] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2} \ \geq\ \frac{3(ab+bc+ca)}{a+b+c}\,.$$

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

It is well known (and easy to prove) that $(a+b+c)^2 \geq 3(ab+bc+ca)$, with equality if and only if $a=b=c$. Thus, it suffices to prove the sharper inequality

$$\sum_{\text{cyclic}} \frac{a^3}{b^2-bc+c^2} \ \geq\ a+b+c\,, \tag{1}$$

which we can rewrite in the following equivalent forms:

$$\sum_{\text{cyclic}} \frac{a^3(b+c)}{b^3+c^3} \ \geq\ a+b+c\,,$$

$$\sum_{\text{cyclic}} a^3(b+c)(c^3+a^3)(a^3+b^3) \ \geq\ (a+b+c)\prod_{\text{cyclic}}(b^3+c^3)\,.$$

We have

$$\begin{aligned}
&\sum_{\text{cyclic}} a^3(b+c)(c^3+a^3)(a^3+b^3) - (a+b+c)\prod_{\text{cyclic}}(b^3+c^3)\\
&\quad= \sum_{\text{cyclic}} a^7b(a^2-b^2) + \sum_{\text{cyclic}} ab^7(b^2-a^2)\\
&\quad= \sum_{\text{cyclic}} (a^7b-ab^7)(a^2-b^2) \ =\ \sum_{\text{cyclic}} ab(a^6-b^6)(a^2-b^2)\\
&\quad= \sum_{\text{cyclic}} ab(a^2-b^2)^2(a^4+a^2b^2+b^4) \ \geq\ 0\,,
\end{aligned}$$

with equality if and only if $a=b=c$. This completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE BALOGLOU, SUNY, Oswego, NY, USA; MICHEL BATAILLE, Rouen, France; VASILE CÎRTOAJE, University of Ploiesti, Romania; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain;

BOGDAN IONIŢĂ, Bucharest, Romania, and TITU ZVONARU, Cománeşti, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; JASNA VILIĆ, Second High School Sarajevo, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Cománeşti, Romania; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

Many solvers used Schur's or Muirhead's Inequality. Most solvers proved the sharper inequality (1). *Cîrtoaje had published this inequality in the Romanian journal Gazeta Matematica, 1(2004), p. 43. Zvonaru and Ioniţă mentioned that the original problem was proposed by Milorad Stefanovic from Yugoslavia (but not used) at the Balkan Mathematical Olympiad 1990. They give a reference to Gazeta Matematica, 5(1991). Woo proved the generalization*

$$\sum_{\text{cyclic}} \frac{a^k(b+c)}{b^k+c^k} \;\geq\; a+b+c$$

for all real $k \geq 1$. *Janous supplied a chain of generalizations. First he proved that*

$$\sum_{\text{cyclic}} \frac{a^3(b+c)}{b^3+c^3} \;\geq\; 2\sum_{\text{cyclic}} \frac{a^3}{b^2+c^2} \;\geq\; a+b+c\,.$$

He then extended the sharper inequality $\sum\limits_{\text{cyclic}} \frac{a^3}{b^2+c^2} \geq \frac{a+b+c}{2}$ *by replacing the left side by* $\sum\limits_{\text{cyclic}} \frac{a^{\lambda+1}}{b^\lambda+c^\lambda}$, *where* $\lambda \geq 0$. *Finally, he suggested a generalization to* n *variables.*

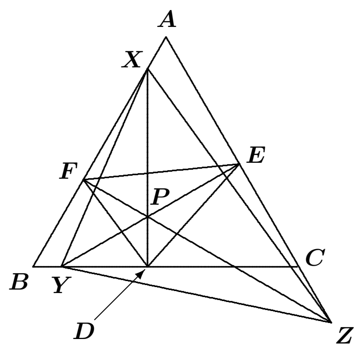
2928. [2004 : 172, 175] *Proposed by Christopher J. Bradley, Bristol, UK.*

Suppose that ABC is an equilateral triangle and that P is a point in the plane of $\triangle ABC$. The perpendicular from P to BC meets AB at X, the perpendicular from P to CA meets BC at Y, and the perpendicular from P to AB meets CA at Z.

1. If P is in the interior of $\triangle ABC$, prove that $[XYZ] \leq [ABC]$.
2. If P lies on the circumcircle of ABC, prove that X, Y, and Z are collinear.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

1. Let D, E, F be the points of intersection of PX with BC, PY with CA, PZ with AB, respectively. Since $XD \perp BC$ and $\angle ABC = 60^\circ$, we have $\angle PXF = \angle DXB = 30^\circ$. Thus, $PF = PX \sin 30^\circ = \frac{1}{2}PX$; that is, $PX = 2PF$. Similarly, $PY = 2PD$ and $PZ = 2PE$.

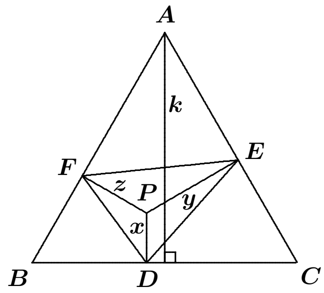


Since $\angle DPZ = \angle XPF = 60^\circ$, $\angle EPX = \angle YPD = 60^\circ$, and $\angle FPY = \angle ZPE = 60^\circ$, we have $\angle XPY = \angle FPD = 120^\circ$. Hence,

$$\frac{[XPY]}{[FPD]} = \frac{PX \cdot PY}{PF \cdot PD} = \frac{2PF \cdot 2PD}{PF \cdot PD} = 4\,.$$

Therefore, $[XPY] = 4[FPD]$. Similarly, we have $[YPZ] = 4[DPE]$ and $[ZPX] = 4[EPF]$. Thus,

$$\begin{aligned}[XYZ] &= [XPY] + [YPZ] + [ZPX] \\ &= 4\big([FPD] + [DPE] + [EPF]\big) \\ &= 4[DEF]\,.\end{aligned}$$



Now let $PD = x$, $PE = y$, and $PF = z$, and let k be the altitude of $\triangle ABC$. As is well known, we have $x + y + z = k$. Since

$$\angle FPD = \angle DPE = \angle EPF = 120^\circ\,,$$

we have

$$\begin{aligned}[DEF] &= [DPE] + [EPF] + [FPD] \\ &= \tfrac{1}{2}xy\sin 120^\circ + \tfrac{1}{2}yz\sin 120^\circ + \tfrac{1}{2}zx\sin 120^\circ \\ &= \tfrac{\sqrt{3}}{4}(xy + yz + zx)\,.\end{aligned}$$

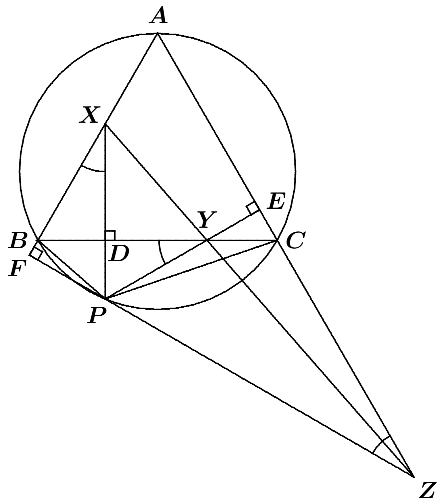
Since

$$(x + y + z)^2 - 3(xy + yz + zx) = x^2 + y^2 + z^2 - (xy + yz + zx) \ge 0\,,$$

we see that $xy + yz + zx \le \frac{1}{3}k^2$. Thus,

$$[DEF] \le \tfrac{\sqrt{3}}{4} \cdot \tfrac{1}{3}k^2 = \tfrac{\sqrt{3}}{12}k \cdot k = \tfrac{\sqrt{3}}{12}k \cdot \tfrac{\sqrt{3}}{2}BC = \tfrac{1}{4}[ABC]\,.$$

Therefore, $[XYZ] \le [ABC]$.



2. Without loss of generality, we may assume that P is a point on the minor arc BC. Since $XD \perp BC$, we have $\angle PXB = \angle DXB = 30^\circ$. Similarly, $\angle PYB = \angle EYC = 30^\circ$ and $\angle PZC = \angle FZA = 30^\circ$. Since $\angle PXB = \angle PYB$, we see that B, P, Y, X are concyclic. Thus,

$$\angle PYX = \angle PBF\,. \qquad (1)$$

Since $\angle PYB = \angle PZC$, we see that P, Z, C, Y are concyclic. Thus,

$$\angle PYZ = \angle PCZ\,. \qquad (2)$$

Since A, B, P, C are concyclic, we have $\angle PBF = \angle PCA$. Hence, from (1), $\angle PYX = \angle PCA$. Using (2), we get

$$\angle PYX + \angle PYZ \;=\; \angle PCA + \angle PCZ \;=\; 180^\circ\,.$$

Therefore, X, Y, Z are collinear.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

1. Assume that $\triangle ABC$ has side-length 2, let G be the centroid of $\triangle ABC$, and let Ω be the disk centred at G with radius $2\sqrt{\frac{2}{3}}$. We will prove the stronger result that $[XYZ] \le [ABC]$ if and only if $P \in \Omega$.

Indeed, set up the coordinate system so that $A = (0,\sqrt{3})$, $B = (1,0)$, and $C = (-1,0)$. Suppose that $P = (u,v)$. Then it is routine to derive that

$$X \;=\; \left(u,\sqrt{3}(1-u)\right)\,, \qquad Y \;=\; \left(u+\sqrt{3}v,0\right)\,,$$
$$Z \;=\; \left(-\tfrac{1}{2}(u-\sqrt{3}v+3),-\tfrac{\sqrt{3}}{2}(u-\sqrt{3}v+1)\right)\,.$$

Thus,

$$\begin{aligned}[XYZ] &= \tfrac{1}{2}\left|\det\begin{pmatrix} u & \sqrt{3}(1-u) & 1 \\ u+\sqrt{3}v & 0 & 1 \\ -\tfrac{1}{2}(u-\sqrt{3}v+3) & -\tfrac{\sqrt{3}}{2}(u-\sqrt{3}v+1) & 1\end{pmatrix}\right| \\ &= \tfrac{3\sqrt{3}}{4}\left|u^2+\left(v-\tfrac{1}{\sqrt{3}}\right)^2-\tfrac{4}{3}\right|\,.\end{aligned}$$

Since $\partial\Omega$ (the boundary of Ω) has the equation $x^2+\left(y-\frac{1}{\sqrt{3}}\right)^2 = \frac{8}{3}$, we see that $[XYZ] \le \sqrt{3} = [ABC]$ if and and only if $P \in \Omega$. Also, equality holds if and only if $P = G$ or $P \in \Omega$.

2. If P lies on the circumcircle of $\triangle ABC$, then $u^2+\left(v-\frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}$. Hence, $[XYZ] = 0$; that is, X, Y, Z are collinear.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2929. [2004 : 172, 175] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Suppose that $\triangle ABC$ has $\angle A = 90^\circ$ and $\angle B > \angle C$. Let H be the foot of the perpendicular from A to BC. The point B' lies on BC and is the mirror image of B in the line AH. Suppose that D is the foot of the perpendicular from B' to AC, that E is the foot of the perpendicular from D to BC, that F is the foot of the perpendicular from B to AB', and that G is the foot of the perpendicular from F to BC. Prove that $AH = DE + FG$.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Since corresponding sides are parallel, we get $\triangle DB'C \sim \triangle ABC$; and since $\angle BB'A = \angle B'BA$, we also have $\triangle FB'B \sim \triangle ABC$. Since AH, DE, FG are corresponding altitudes of $\triangle ABC$, $\triangle DB'C$, $\triangle FB'B$, respectively, we have

$$\frac{DE}{AH} + \frac{FG}{AH} \;=\; \frac{B'C}{BC} + \frac{B'B}{BC} \;=\; 1\,,$$

and the result follows.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; D. KIPP JOHNSON, Beaverton, OR, USA; DOUG NEWMAN, Lancaster, CA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; TOSHIO SEIMIYA, Kawasaki, Japan; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; JASNA VILIĆ, Second High School Sarajevo, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2930. [2003 : 173, 175] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Suppose that a, b, and c are positive real numbers. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - 27\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right)^{-2} \;\ge\; \frac{1}{3}\left[\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \left(\frac{1}{b} - \frac{1}{c}\right)^2 + \left(\frac{1}{c} - \frac{1}{a}\right)^2\right].$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Expanding the right side and simplifying, we see that the inequality is equivalent to

$$\frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 \;\ge\; 27\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ac}{b}\right)^{-2}\;;$$

that is,

$$\left(\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3}\right)^2 \left(\frac{\frac{ab}{c} + \frac{bc}{a} + \frac{ac}{b}}{3}\right)^2 \;\ge\; 1\,,$$

which is clearly true by the AM–GM Inequality.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE T. BAILEY, ELSIE M. CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, *Beaverton, OR, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; JASNA VILIĆ, Second High School Sarajevo, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.*

Bencze poses the following generalization. If $x_k > 0$ for $k = 1, 2, \ldots, n$, then

$$\left(\sum_{k=1}^{n} \frac{1}{x_k^{2p}}\right)^{m-2p} - n^m \left(\sum_{k=1}^{m} x_1 \cdots x_{k-1} x_k^{1-2p} x_{k+1} \cdots x_n\right)^{-2p} \geq \frac{1}{n} \sum_{1 \leq i < j \leq n} \left(\frac{1}{x_i} - \frac{1}{x_j}\right)^{2p}$$

2931. [2004 : 173, 175] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

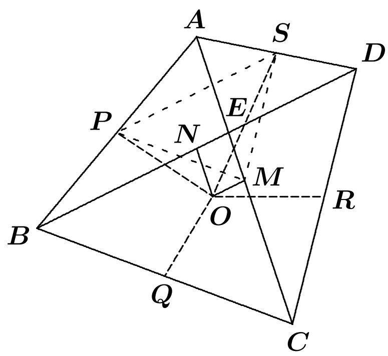
Given quadrilateral $ABCD$, let P, Q, R, S, M and N be the midpoints of AB, BC, CD, DA, AC and BD, respectively. Suppose that the diagonals AC and BD intersect at E. Let O be the point such that quadrilateral $NEMO$ is a parallelogram.

Prove that $[OPAS] = [OQBP] = [ORCQ] = [OSDR]$ (where $[WXYZ]$ represents the area of quadrilateral $WXYZ$.)

Solution by Joel Schlosberg, Bayside, NY, USA. [Ed.: *Very similar solutions were submitted by D. Kipp Johnson, Beaverton, OR, USA; Toshio Seimiya, Kawasaki, Japan; Peter Y. Woo, Biola University, La Mirada, CA, USA; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.*]

A dilation with centre A and factor 2 takes PS onto BD and $MPAS$ onto $CBAD$. It follows that $PS \parallel BD$ and $[CBAD] = 4[MPAS]$. Since $MO \parallel BD$ and $BD \parallel PS$, we get $MO \parallel PS$. Hence, $[OPS] = [MPS]$ and

$$[OPAS] = [MPAS] = \tfrac{1}{4}[ABCD].$$



The same reasoning shows that

$$[OQBP] = [NQBP] = \tfrac{1}{4}[ABCD],$$
$$[ORCQ] = [MRCQ] = \tfrac{1}{4}[ABCD],$$
$$[OSDR] = [NSDR] = \tfrac{1}{4}[ABCD],$$

giving the desired result.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DOUG NEWMAN, Lancaster, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and TITU ZVONARU, Comănești, Romania.

2932. [2004 : 173, 175] *Proposed by Titu Zvonaru, Bucharest, Romania.*

In $\triangle ABC$, suppose that the points M, N lie on the line segment BC, the point P lies on the line segment CA, and the point Q lies on the line segment AB, such that $MNPQ$ is a square. Suppose further that

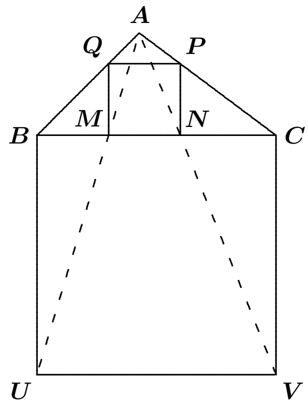
$$\frac{AM}{AN} = \frac{AC+\sqrt{2}AB}{AB+\sqrt{2}AC}.$$

Characterize $\triangle ABC$.

Solution by Michel Bataille, Rouen, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

We will prove that $\frac{AM}{AN} = \frac{AC+\sqrt{2}AB}{AB+\sqrt{2}AC}$ if and only if $AB = AC$ or $\angle BAC = \frac{3\pi}{4}$.

Let $a = BC$, $b = CA$, and $c = AB$. Consider the homothety $h\left(A, \frac{AB}{AQ}\right)$. Let U and V be the images of points M and N, respectively. Then $BUVC$ is the image of the square $QMNP$, and thus, it is also a square. Hence, $BU = CV = a$.



Using the Laws of Cosines and Sines, and the trigonometric identity

$$\sin A - \cos A = \sqrt{2}\sin\left(A - \frac{\pi}{4}\right),$$

we obtain

$$\begin{aligned} AU^2 &= c^2 + a^2 - 2ac\cos\left(B + \frac{\pi}{2}\right) \\ &= c^2 + a^2 + 2ac\sin B \\ &= c^2 + a^2 + 2bc\sin A \\ &= c^2 + (b^2 + c^2 - 2bc\cos A) + 2bc\sin A \\ &= 2c^2 + b^2 + 2bc(\sin A - \cos A) \\ &= \left(b + \sqrt{2}c\right)^2 + 2\sqrt{2}bc\left(\sin\left(A - \frac{\pi}{4}\right) - 1\right). \end{aligned}$$

Thus, $AU^2 = \left(b+\sqrt{2}c\right)^2 + m$, where $m = 2\sqrt{2}bc\left(\sin\left(A - \frac{\pi}{4}\right) - 1\right)$. Similarly, $AV^2 = \left(c+\sqrt{2}b\right)^2 + m$. Since the homothety implies that $\frac{AM}{AN} = \frac{AU}{AV}$, the given condition is equivalent to

$$\frac{\left(b+\sqrt{2}c\right)^2}{\left(c+\sqrt{2}b\right)^2} \;=\; \frac{\left(b+\sqrt{2}c\right)^2+m}{\left(c+\sqrt{2}b\right)^2+m}\,.$$

This simplifies to $m(b^2 - c^2) = 0$, which is true if and only if $b^2 = c^2$ or $\sin\left(A - \frac{\pi}{4}\right) = 1$; that is, $b = c$ or $A = \frac{3\pi}{4}$.

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; D.J. SMEENK, Zaltbommel, the Netherlands (2 solutions); and the proposer. There were also two incomplete solutions.

2933. [2004 : 173, 176] *Proposed by Titu Zvonaru, Bucharest, Romania.*

Prove, without the use of a calculator, that $\sin(40^\circ) < \sqrt{\frac{3}{7}}$.

I. *Solution by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela.*

Since

$$\begin{aligned}\sin(40^\circ) &= 2\sin(20^\circ)\cos(20^\circ) \;<\; 2\sin(20^\circ)\\ &= 2\sin(60^\circ - 40^\circ) \;=\; \sqrt{3}\cos(40^\circ) - \sin(40^\circ)\,,\end{aligned}$$

we have $2\sin(40^\circ) < \sqrt{3}\cos(40^\circ)$.

Hence,

$$4\sin^2(40^\circ) \;<\; 3\cos^2(40^\circ) \;=\; 3(1 - \sin^2(40^\circ))\,,$$

or $7\sin^2(40^\circ) < 3$, from which $\sin(40^\circ) < \sqrt{\frac{3}{7}}$ follows immediately.

II. *Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

We prove the slightly stronger inequality $\sin(40^\circ) < \sqrt{\frac{5}{12}}$.

Note first that $\sin(40^\circ) < \sqrt{\frac{5}{12}}$ is equivalent to $\frac{1}{2}\big(1 - \cos(80^\circ)\big) < \frac{5}{12}$, or $\cos(80^\circ) > \frac{1}{6}$, which is the same as $\sin(10^\circ) > \frac{1}{6}$. Let $c = \sin(10^\circ)$. Then $0 < c < 1$. From $\frac{1}{2} = \sin(30^\circ) = 3\sin(10^\circ) - 4\sin^3(10^\circ) = 3c - 4c^3$, we obtain $8c^3 - 6c + 1 = 0$. Since $8c^3 > 0$, we must have $-6c + 1 < 0$. Hence, $c > \frac{1}{6}$, and we are done.

Also solved by SAMUEL ALEXANDER, student, University of Arizona, Tucson, AZ, USA; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines, and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; KEE-WAI LAU, Hong Kong, China; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; DOUG NEWMAN, Lancaster, CA, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; D.J. SMEENK, Zaltbommel, the Netherlands; MIKE SPIVEY, Samford

University, Birmingam, AL, USA; Mª JESÚS VILLAR RUBIO, Santander, Spain; STAN WAGON, Macalester College, St. Paul, MN, USA; MICHAEL WATSON, student, University of Waterloo, Waterloo, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Solvers who obtained the sharper bound given in Solution II above were Bailey et al., Beasley, Hess, and Janous.

Most of the solutions used calculus or concavity and Jensen's Inequality.

Wagon commented "it is pointless to pose a problem with the restriction 'without the use of a calculator' when calculations can be done by pencil and paper. The only meaningful interpretation of the problem is that it was meant to be done without numerical computations of any sort, but this would preclude using $1 + 1 = 2$. So, I have to say that I feel that such problems should not appear in Crux". We invite our readers to send us your opinions on whether you agree with his comments.

2934. [2004 : 173, 176] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$ with circumradius R, let AD, BE and CF be the altitudes. Let P be any interior point of the triangle. The line through P parallel to EF intersects the line AC at E_1 and the line AB at F_1. The line through P parallel to FD intersects the line AB at F_2 and the line BC at D_2. The line through P parallel to DE intersects the line BC at D_3 and the line AC at E_3.

Show that

$$E_1F_1 \cot A + F_2D_2 \cot B + D_3E_3 \cot C \;=\; 2R\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let L, M, and N be the feet of the perpendiculars from P to BC, CA, and AB, respectively. Note that

$$\angle PF_1F_2 \;=\; \angle EFA \;=\; \angle C \;=\; \angle DFB \;=\; \angle PF_2F_1\,;$$

thus, $PF_1 = PF_2$. Likewise, $PD_2 = PD_3$ and $PE_3 = PE_1$. Hence,

$$\begin{aligned}
&E_1F_1 \cot A + F_2D_2 \cot B + D_3E_3 \cot C \\
&= (PF_1 + PE_1)\cot A + (PF_2 + PD_2)\cot B + (PE_3 + PD_3)\cot C \\
&= (PF_1 + PE_3)\cot A + (PF_1 + PD_2)\cot B + (PE_3 + PD_2)\cot C \\
&= PF_1(\cot A + \cot B) + PD_2(\cot B + \cot C) + PE_3(\cot C + \cot A) \\
&= \frac{PF_1 \cdot AB}{CF} + \frac{PD_2 \cdot BC}{AD} + \frac{PE_3 \cdot CA}{BE} \\
&= 2R\left(\frac{PF_1 \cdot \sin C}{CF} + \frac{PD_2 \cdot \sin A}{AD} + \frac{PE_3 \cdot \sin B}{BE}\right) \\
&= 2R\left(\frac{PN}{CF} + \frac{PL}{AD} + \frac{PM}{BE}\right) \;=\; 2R\left(\frac{[APB]}{[ABC]} + \frac{[BPC]}{[ABC]} + \frac{[CPA]}{[ABC]}\right) \\
&= 2R\,.
\end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; D. KIPP JOHNSON, Beaverton, OR, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2935. [2004 : 174] *Proposed by Titu Zvonaru, Bucharest, Romania.*

Suppose that a, b, and c are positive real numbers which satisfy $a^2+b^2+c^2=1$, and that $n>1$ is a positive integer. Prove that

$$\frac{a}{1-a^n}+\frac{b}{1-b^n}+\frac{c}{1-c^n}\;\geq\;\frac{(n+1)^{1+\frac{1}{n}}}{n}\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

For $0<x<1$ we have, by the AM–GM Inequality,

$$\begin{aligned}\big(x(1-x^n)\big)^n &= \frac{nx^n(1-x^n)^n}{n}\;\leq\;\frac{1}{n}\left(\frac{nx^n+n(1-x^n)}{n+1}\right)^{n+1}\\ &= \frac{n^n}{(n+1)^{n+1}}\,,\end{aligned}$$

from which we see that $x(1-x^n)\leq\dfrac{n}{(n+1)^{1+\frac{1}{n}}}$. Hence,

$$\begin{aligned}\sum_{\text{cyclic}}\frac{a}{1-a^n} &= \sum_{\text{cyclic}}\frac{a^2}{a(1-a^n)}\;\geq\;\sum_{\text{cyclic}}\frac{a^2(n+1)^{1+\frac{1}{n}}}{n}\\ &= \frac{(n+1)^{1+\frac{1}{n}}}{n}(a^2+b^2+c^2)\;=\;\frac{(n+1)^{1+\frac{1}{n}}}{n}\,.\end{aligned}$$

Equality holds when $n=2$ and $a=b=c=1/\sqrt{3}$.

Also solved by MICHEL BATAILLE, Rouen, France; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; BABIS STERGIOU, Chalkida, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Both Howard and the proposer remarked that this problem is a generalization of Crux 2738 [2002 : 180; 2003 : 243]. *Heuver remarked that Crux 1445* [1989 : 148; 1990 : 216] *by the late Murray Klamkin and Andy Liu dealt with a more generalized version of this problem. Janous, noticing that the lower bound is not sharp, offered three conjectures, one of which is as follows:* Let x_1, x_2, …, x_n be positive real numbers satisfying $x_1^2+x_2^2+\cdots+x_n^2=1$. Then, for all $p>0$, we have $\displaystyle\sum_{j=1}^{n}\frac{x_j}{1-x_j^p}\geq\frac{n^{(p+1)/2}}{n^{p/2}-1}$.

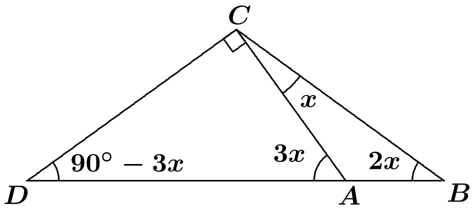
Though a few solvers stated that equality holds in the given inequality only if $n=2$ and $a=b=c=1/\sqrt{3}$, no one actually gave a detailed proof (though this is not difficult). Indeed, from the proof given in the solution above we see that if equality holds, then we must have $nx^n=1-x^n$, or $x=\dfrac{1}{\sqrt[n]{n+1}}$ which implies that $a=b=c=\dfrac{1}{(n+1)^{1/n}}$. From $a^2+b^2+c^2=1$, we then get $\dfrac{3}{(n+1)^{2/n}}=1$ or $(n+1)^2=3^n$, which clearly holds when $n=2$. But by a simple induction, one can easily show that $3^n>(n+1)^2$ for all $n\geq3$, and the conclusion follows.

2936. [2004 : 174] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Consider $\triangle ABC$ with $\angle ABC = 2\angle ACB$ and $\angle BAC > 90^\circ$. Given that the perpendicular to AC through C meets AB at D, prove that

$$\frac{1}{AB} - \frac{1}{BD} = \frac{2}{BC}\,.$$

I. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

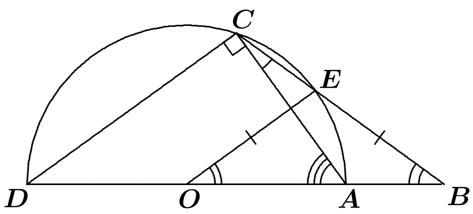


Let $\angle ACB = x$. Then $\angle ABC = 2x$, $\angle DAC = 3x$, $\angle BAC = \pi - 3x$, $\angle BCD = \frac{\pi}{2} + x$, and $\angle BDC = \frac{\pi}{2} - 3x$. Using the Sine Law on triangles ABC and BCD, we obtain

$$\begin{aligned}\frac{BC}{AB} - \frac{BC}{BD} &= \frac{\sin\angle BAC}{\sin\angle ACB} - \frac{\sin\angle BDC}{\sin\angle BCD} = \frac{\sin(\pi - 3x)}{\sin x} - \frac{\sin(\frac{\pi}{2} - 3x)}{\sin(\frac{\pi}{2} + x)} \\ &= \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = \frac{\sin 3x\cos x - \cos 3x\sin x}{\sin x\cos x} \\ &= \frac{\sin(3x - x)}{\frac{1}{2}\sin 2x} = 2\,,\end{aligned}$$

and the result follows.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*



Let O be the mid-point of the segment AD. Since $\angle ACD = 90^\circ$, the circumcentre of $\triangle ACD$ is at O. Suppose the circumcircle of $\triangle ACD$ intersects BC at E. Then $\angle AOE = 2\angle ACE = \angle ABE$. Hence, $BE = OE = \frac{1}{2}AD$. Therefore,

$$\frac{1}{2}AD \cdot BC = BE \cdot BC = BA \cdot BD\,,$$

so that

$$\frac{2AB}{BC} = \frac{AD}{BD} = \frac{BD - AB}{BD} = 1 - \frac{AB}{BD},$$

which gives

$$\frac{1}{AB} - \frac{1}{BD} = \frac{2}{BC},$$

as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; RICARDO BARROSO CAMPOS, Universidad de Sevilla, Sevilla, Spain; MARLON CAUMERAN, Philippine Science High School, Central Mindanao Campus, and I.J.L. GARCES, Ateneo de Manila University, The Phlippines; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOSÉMARÍA PEDRET, Barcelona, Spain; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. There was also one incorrect solution submitted.

Zhou and Pedret were the only solvers who submitted a solution that did not use trigonometry.

2937. [2004 : 174, 176] *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Suppose that $x_1, \ldots, x_n$ ($n \geq 2$) are positive real numbers. Prove that

$$\left(x_1^2 + \cdots + x_n^2\right)\left(\frac{1}{x_1^2 + x_1x_2} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right) \geq \frac{n^2}{2}.$$

I. *Composite of essentially the same solution by Arkady Alt, San Jose, CA, USA; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; John G. Heuver, Grande Prairie, AB; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Since

$$\begin{aligned} &2(x_1^2 + x_2^2 + \cdots + x_n^2) - 2(x_1x_2 + x_2x_3 + \cdots + x_nx_1) \\ &\qquad = (x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_n - x_1)^2 \geq 0, \end{aligned}$$

we have $x_1^2 + x_2^2 + \cdots + x_n^2 \geq x_1x_2 + x_2x_3 + \cdots + x_nx_1$. Hence,

$$\begin{aligned} &2(x_1^2 + x_2^2 + \cdots + x_n^2)\left(\frac{1}{x_1^2 + x_1x_2} + \frac{1}{x_2^2 + x_2x_3} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right) \\ &\qquad \geq \Big((x_1^2 + x_1x_2) + (x_2^2 + x_2x_3) + \cdots + (x_n^2 + x_nx_1)\Big) \cdot \\ &\qquad\qquad \cdot \left(\frac{1}{x_1^2 + x_1x_2} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right). \end{aligned}$$

The right side is at least n^2, by the AM–HM Inequality.

Clearly, equality holds if and only if all the x_i's are equal.

II. *Generalization by Mihály Bencze, Brasov, Romania, adapted by the editor.*

We show more generally that if $x_k > 0$, $k = 1, 2, \ldots, n$, where $n \geq 3$, then for all a, $b > 0$ and for all m with $2 \leq m \leq n-1$ we have

$$\left(\sum_{k=1}^{n} x_k^m\right) \sum_{k=1}^{n} \frac{1}{ax_k^m + bx_k x_{k+1} \cdots x_{k+m-1}} \geq \frac{n^2}{a+b}, \qquad (1)$$

where all indices are reduced modulo n. The current problem is the special case when $m = 2$ and $a = b = 1$.

To prove (1), we first apply the Power–Mean Inequality to obtain

$$\begin{aligned} \sum_{k=1}^{n} ax_k^m + \sum_{k=1}^{n} bx_k x_{k+1} \cdots x_{k+m-1} & \\ \leq \; & a\sum_{k=1}^{n} x_k^m + b\sum_{k=1}^{n} \frac{x_k^m + x_{k+1}^m + \cdots + x_{k+m-1}^m}{m} \\ = \; & (a+b)\sum_{k=1}^{n} x_k^m \,. \qquad (2) \end{aligned}$$

Next we have, by the AM–HM Inequality,

$$\begin{aligned} \left(\sum_{k=1}^{n} (ax_k^m + bx_k x_{k+1} \cdots x_{k+m-1})\right) \cdot & \\ \cdot \sum_{k=1}^{n} \frac{1}{ax_k^m + bx_k x_{k+1} \cdots x_{k+m-1}} \; & \geq \; n^2 \qquad (3) \end{aligned}$$

From (2) and (3), we see that (1) follows immediately.

Also solved by MICHEL BATAILLE, Rouen, France; VASILE CÎRTOAJE, University of Ploiesti, Romania; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incomplete solution.

Most of the solutions use one or more of the following: AM–GM Inequality; AM–HM Inequality, Cauchy–Schwarz Inequality, Power–Mean Inequality, homogeneity, and convexity.

Cîrtoaje conjectured the following sharper inequality and proved it to be true for $n = 3$ *and* $n = 4$*:*

$$(x_1x_2 + x_2x_3 + \cdots + x_nx_1)\left(\frac{1}{x_1^2 + x_1x_2} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right) \geq \frac{n^2}{2}\,.$$

Janous also obtained a generalization which is the special case when $a = b = 1$ *of Bencze's generalization featured above.*

2938. [2004 : 174, 176, 296, 298] *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Suppose that $x_1, \ldots, x_n, \alpha$ are positive real numbers. Prove that

(a) $\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha)} \;\geq\; \alpha + \sqrt[n]{x_1\cdots x_n}$;

(b) $\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha)} \;\leq\; \alpha + \dfrac{x_1+\cdots+x_n}{n}$.

Obervation by Michel Bataille, Rouen, France; Vedula N. Murty, Dover, PA, USA; and Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.

Part (a) is a special case of 2176 [1996 : 275; 1997 : 444] (which also appeared as 2730 [2002 : 117; 2003 : 186]) with $a_1 = \cdots = a_n = \alpha$ and $b_1 = x_1, \ldots, b_n = x_n$.

Part (b) follows immediately from the AM–GM Inequality applied to the positive real numbers $x_1+\alpha, \ldots, x_n+\alpha$.

Full solutions of the problem were submitted by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D. KIPP JOHNSON, Beaverton, OR, USA; VINAYAK MURALIDHAR, student, Corona del Sol High School, Tempe, AZ, USA; TREY SMITH, Angelo State University, San Angelo, TX, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Furdui notes that the problem is an application of problem A2 on the 64th Putnam Competition.

Mihály Bencze, Brasov, Romania provided many extended results from this inequality. The interested reader is referred to "Mihály Bencze, About M. Bencze's Polynomial Inequalities, Octogon Mathematical Magazine, **9** *(2001) No. 1, pp. 263–272."*