Iterating Möbius Functions with Rational Coefficients, Part I

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We will determine all the possible periods of a periodic sequence of functions obtained by iterating a Möbius function with rational coefficients.

1. Introduction

A *Möbius function* is a function of the form

$$f(z) \;=\; \frac{az+b}{cz+d}\,, \tag{1}$$

where z is a complex variable and the coefficients a, b, c, and d are complex numbers such that $ad \neq bc$. The coefficients form a 2×2 matrix

$$A \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

This matrix is not uniquely determined by f; it can be multiplied by any non-zero complex number. The *determinant* of the matrix A is defined to be $\det(A) = ad - bc$. Our assumption that $ad \neq bc$ is equivalent to the condition $\det(A) \neq 0$. In other words, the matrix A is supposed to be invertible.

Note that the identity function $f(z) = z$ is a Möbius function whose coefficient matrix is the 2×2 *identity matrix* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Let f and g be Möbius functions with respective coefficient matrices

$$A \;=\; \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \quad \text{and} \quad B \;=\; \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}.$$

The composite function $f \circ g$ is defined by $f \circ g(z) = f\big(g(z)\big)$. Thus,

$$f \circ g(z) \;=\; \frac{a_1\left(\dfrac{a_2z+b_2}{c_2z+d_2}\right)+b_1}{c_1\left(\dfrac{a_2z+b_2}{c_2z+d_2}\right)+d_1} \;=\; \frac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)}\,.$$

We see that $f \circ g$ is another Möbius function, and that its coefficient matrix is the matrix product AB. (As a composite function, $f \circ g$ may be undefined at certain points, but we will always assume that continuous extensions have been made wherever possible to maximize the domain. This will allow us to say, for example, that if $f(z) = 1/z$, then $f \circ f$ is the identity function.)

Given a Möbius function f, we define a sequence $\{f_k\}_{k=0}^{\infty}$ of Möbius functions, where f_0 is the identity function, $f_1 = f$, and $f_k = f \circ f_{k-1}$ for $k \geq 2$. It is easy to prove inductively that the coefficient matrix of f_k is A^k, where A is the coefficient matrix of f.

The sequence $\{f_k\}$ is said to be *periodic* if there is a positive integer n such that $f_n = f_0$. The smallest such integer n is then called the *period* of the sequence. Thus, if n is the period, there is no positive integer $m < n$ for which $f_m = f_0$. Actually, we need only rule out the positive divisors m of n. If ℓ is another positive integer for which $f_\ell = f_0$, then the greatest common divisor of n and ℓ also has this property.

From now on, we assume that the coefficients of f are rational. Then each function f_k has rational coefficients. The sequence $\{f_k\}$ is periodic if and only if $A^n = \alpha I$ for some positive integer n and some non-zero rational number α. Moreover, if $A^m \neq \beta I$ for any positive divisor m of n and any non-zero rational number β, then the sequence is of period n. We will determine all the possible periods.

2. Case Studies

The identity function $f(z) = z$ is periodic of period 1, and it is the only function with this property.

To find examples with a period of 2, we calculate

$$A^2 = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix}.$$

We see that A^2 is a rational multiple of I if $a + d = 0$. Two obvious examples are $f(z) = -z$ and $f(z) = 1/z$.

To simplify subsequent computations, we note that, if $c = 0$, then $d \neq 0$, and f is a linear function. It is easy to check that a linear function (with rational coefficients) cannot generate a periodic sequence of functions of period greater than 2. Henceforth, we assume that $c \neq 0$. Without loss of generality, we may assume that $c = 1$.

Let $g(z) = z - d$. The inverse function for g is $g^{-1}(z) = z + d$. Define $h(z) = g^{-1}(f(g(z)))$. Let B and C be the coefficient matricies for g and h respectively. Then $B^{-1}AB = C$. Explicitly,

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 1 & d \end{bmatrix} \begin{bmatrix} 1 & -d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a+d & b-ad \\ 1 & 0 \end{bmatrix}.$$

We draw the attention of the reader to the form of the bottom row of C.

Since $C^k = (B^{-1}AB)^k = B^{-1}A^kB$, we see that C^k is a rational multiple of I if and only if A^k is. Therefore, if $\{f_k\}$ is a periodic sequence, then $\{h_k\}$ is a periodic sequence with the same period. Thus, we can replace f by h. It follows that we may assume not only $c = 1$, but also $d = 0$. This may cause some examples of periodic sequences of a certain period to be lost, but we cannot lose them all. Since all we need is one example for each possible period, our assumption will cause no problem. Henceforth, we take $A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$.

For $n = 3$, we have

$$A^3 = \begin{bmatrix} a^3 + 2ab & b(a^2 + b) \\ a^2 + b & ab \end{bmatrix}.$$

If this is a rational multiple of I, we must have $a^2 + b = 0$. Then $A^3 = -a^3 I$. Taking $a = 1$, we obtain the following example of a function for which $\{f_k\}$ has period 3:

$$f(z) = \frac{z - 1}{z}.$$

For $n = 4$, we have

$$A^4 = \begin{bmatrix} a^4 + 3a^2b + b^2 & ab(a^2 + 2b) \\ a(a^2 + 2b) & b(a^2 + b) \end{bmatrix}.$$

If this is a rational multiple of I, we must have $a(a^2 + 2b) = 0$. However, $a = 0$ leads to a sequence of period 2. Hence, we must have $a^2 + 2b = 0$. Then $A^4 = -\frac{a^4}{4} I$. Taking $a = 1$, we obtain an example of a function for which $\{f_k\}$ has period 4:

$$f(z) = \frac{2z - 1}{2z}.$$

It looks as if all values might be possible, but now we come up against the first negative case.

For $n = 5$, we have

$$A^5 = \begin{bmatrix} a^5 + 4a^3b + 3ab^2 & b(a^4 + 3a^2b + b^2) \\ a^4 + 3a^2b + b^2 & ab(a^2 + 2b) \end{bmatrix}.$$

If this is a rational multiple of I, we must have $a^4 + 3a^2b + b^2 = 0$. This is a quadratic equation in a^2/b. It follows easily from the Quadratic Formula that there are no rational solutions.

For $n = 6$, we have

$$A^6 = \begin{bmatrix} a^6 + 4a^4b + 4a^2b^2 + b^3 & ab(a^2 + b)(a^2 + 3b) \\ a(a^2 + b)(a^2 + 3b) & a^4b + 3a^2b^2 + b^3 \end{bmatrix}.$$

If this is a rational multiple of I, we must have $a(a^2 + b)(a^2 + 3b) = 0$. However, $a = 0$ leads to a sequence of period 2 while $a^2 + b = 0$ leads to a sequence of period 3. Hence, $a^2 + 3b = 0$. Then $A^4 = -\frac{a^4}{4} I$. Taking $a = 1$, we obtain a function for which $\{f_k\}$ has period 6:

$$f(z) = \frac{3z - 1}{3z}.$$

The next case is $n = 7$, but we suspect that, like the case $n = 5$, it would not work. Thus, we will skip over it.

For $n = 8$, we have

$$A^8 = \begin{bmatrix} a^8 + 7a^6b + 15a^4b^2 + 10a^2b^3 + b^4 & ab(a^2+2b)(a^4+4a^2b+2b^2) \\ a(a^2+2b)(a^4+4a^2b+2b^2) & a^6b + 5a^4b^2 + 6a^2b^3 + b^4 \end{bmatrix}.$$

If this is a rational multiple of I, we must have

$$a(a^2+2b)(a^4+4a^2b+2b^2) = 0.$$

However, $a = 0$ leads to a sequence of period 2 while $a^2 + 2b = 0$ leads to a sequence of period 4. Hence, $a^4 + 4a^2b + 2b^2 = 0$. This is a quadratic equation in a^2/b. It follows easily from the Quadratic Formula that there are no rational solutions. Thus, the case $n = 8$ does not work either.

We now skip ahead to a case that is more likely to work than those passed over. For $n = 12$, we have

$$A^{12} = \begin{bmatrix} \begin{matrix} a^{12} + 11a^{10}b + 45a^8b^2 + 84a^6b^3 \\ +70a^4b^4 + 19a^2b^5 + b^6 \end{matrix} & \begin{matrix} ab(a^2+b)(a^2+2b)(a^2+3b) \\ (a^4+4a^2b+b^2) \end{matrix} \\ \begin{matrix} a(a^2+b)(a^2+2b)(a^2+3b) \\ (a^4+4a^2b+b^2) \end{matrix} & \begin{matrix} a^{10}b + 9a^8b^2 + 28a^6b^3 \\ +35a^4b^4 + 15a^2b^5 + b^6 \end{matrix} \end{bmatrix}.$$

If this is a rational multiple of I, we must have

$$a(a^2+b)(a^2+2b)(a^2+3b)(a^4+4a^2b+b^2) = 0.$$

However, $a = 0$ leads to a sequence of period 2, $a^2 + b = 0$ leads to a sequence of period 3, $a^2 + 2b = 0$ leads to a sequence of period 4, and $a^2 + 3b = 0$ leads to a sequence of period 6. Hence, $a^4 + 4a^2b + b^2 = 0$. This is a quadratic equation in a^2/b. It follows easily from the Quadratic Formula that there are no rational solutions.

At this point, we make a bold conjecture that the only possible values for the period of a periodic sequence $\{f_k\}$ are 1, 2, 3, 4, and 6. In Part II, we will prove this conjecture.

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