THE OLYMPIAD CORNER

No. 244

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We start by giving the Compositions de Mathématiques of the Concours Général des Lyceés, 2001, Classe terminale S. Again my thanks go to Chris Small, Canadian Team Leader to the 42^{nd} IMO, for obtaining them for us.

CONCOURS GÉNÉRAL DES LYCÉES Session de 2001
COMPOSITION DE MATHÉMATIQUES
Class terminale S – Durée : 5 heures

Les premières questions de chacune des quatre parties de ce problème sont indépendantes des autres parties. Il n'est donc pas obligatoire de commencer son étude dans l'ordre indiqué. Les candidats peuvent admettre les résultats d'une question, à condition de l'indiquer clairement sur la copie.

On appelle *trio* tout triplet de nombres réels (a, b, c) non tous nuls et vérifiant la relation :

$$ab + bc + ca = 0 .$$

Lorsque $a + b + c = 1$, on dit que le trio (a, b, c) est un trio *réduit*.

Les coordonnées sont rapportées à un repère orthonormal direct $(0, \overrightarrow{I}, \overrightarrow{J}, \overrightarrow{K})$ de l'espace.

Première partie

On note C l'ensemble des points de coordonnées (a, b, c) où (a, b, c) est un trio. On note Γ l'ensemble des points de coordonnées où (a, b, c) est un trio réduit. On note $\mathcal{P}$ le plan d'équation $x + y + z = 1$.

1. Existe-t-il des trios (a, b, c) tels que $a + b + c = 0$?

2. Montrer que C est une réunion de droites passant par O et privées de ce point.

3. Montrer que Γ est l'intersection d'un plan et d'une sphère de centre O. Quelle est la nature géométrique de Γ?

4. Donner la nature géométrique de C et l'illustrer par un croquis.

5. Soit L un point fixé de Γ. Montrer que le volume V du tétrèdre $OLL'L''$, où L' et L'' sont deux points distincts de Γ et différents de L, est maximal lorsque les arêtes issues de O sont deux à deux orthogonales et déterminer alors les coordonnées de L' et L'' en fonction de celles de L.

6. Montrer que le produit abc admet un maximum et un minimum lorsque le point de coordonnées (a, b, c) décrit Γ. Préciser les trios réduits réalisant ces extrémums.

Deuxième partie

Dans cette partie et les suivantes, un trio (a, b, c) est dit *rationnel* lorsque a, b et c sont des ombres rationnels (éléments de l'ensemble $\mathbb{Q}$) ; il est dit *entier* lorsque a, b et c sont des nombres entiers relatifs (éléments de l'ensemble $\mathbb{Z}$) ; enfin un trio entier est dit *primitif* si a, b et c n'admettent que 1 et -1 comme diviseurs communs.

1. Déterminer la nature de l'ensemble H_1 des points de coordonnées $(x, y, 1)$ tels que $(x, y, 1)$ soit un trio. Montrer que le point Ω_1 de coordonnées $(-1, -1, 1)$ est un centre de symétrie de H_1. Quels sont les points de H_1 à coordonnées entières ?

2. Pour tout entier naturel non nul h, on note Z_h l'ensemble des trois entiers (a, b, c) tels que $c = h$. Déterminer Z_h pour $h = 1$ et $h = 2$.

3. Montrer que Z_h est un ensemble fini et exprimer le nombre $H(h)$ de ses éléments en fonction de celui des diviseurs de h^2 dans $\mathbb{Z}$. Montrer que 4 divise $N(h) - 2$.

4. Pour tout entier naturel non nul h, on note $N'(h)$ le nombre de trios entiers (a, b, c) tels que l'un au moins des entiers a, b ou c soit égal à h. Exprimer $N'(h)$ en fonction de $N(h)$ selon la parité de h.

5. Montrer qu'à tout trio entier (a, b, c) on peut associer un triplet (r, s, t) d'entiers tels que r et s soient premiers entre eux, s positif ou nul, et tels que l'on ait :

$$a = r(r + s)t\,, \quad b = s(r + s)t\,, \quad c = -rst\,.$$

Énoncer et démontrer une réciproque. Pour quels trios (a, b, c) le triplet (r, s, t) n'est-il pas unique ?

6. Déterminer les triplets (r, s, t) ainsi associés aux trios primitifs. En déduire que si (a, b, c) est un trio primitif, alors $|abc|$, $|a + b|$, $|c|$ et $|c + a|$ sont des carrés d'entiers.

7. Pour tout entier naturel non nul h, on note $P(h)$ le nombre de trios primitifs (a, b, c) tels que $c = h$. Montrer que $P(h)$ est une puissance de 2. Pour quels entiers h a-t-on $P(h) = N(h)$? Expliciter une suite d'entiers (h_n) telle que la suite $\big(P(h_n)/N(h_n)\big)$ converge vers zéro.

8. Soit $(a, b, 1)$ un trio. Montrer qu'il existe deux suites (x_n) et (y_n) convergeant respectivement vers a et b et telles que, pour tout n, $(x_n, y_n, 1)$ soit un trio rationnel.

9. Soit (a, b, c) un trio réduit. Montrer qu'il existe trois suites (x_n), (y_n) et (z_n) convergeant respectivement vers a, b et c et telles que, pour tout n, (x_n, y_n, z_n) soit un trio rationnel réduit.

Troisième partie

On note j le nombre complexe $e^{2i\pi/3}$, c'est-à-dire $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Pour tout trio $T = (a, b, c)$ on note $\widehat{T} = (a, c, b)$, $S(T) = a + b + c$ et $z(T) = a + bj + cj^2$.

1. Calculer le module de $z(T)$ en fonction de $S(T)$. Peut-on avoir $z(T) = 0$? Calculer le cosinus et le sinus d'un argument θ de $z(T)$ en fonction de a, b et c.

2. Soit z_0 un nombre complexe non nul. Déterminer les trios $T = (a, b, c)$ tels que $z(T) = z_0$.

3. Étant donnés deux trios T_1 et T_2, montrer qu'il existe un unique trio, noté $T_1 * T_2$, vérifiant $S(T_1 * T_2) = S(T_1)S(T_2)$ et $z(T_1 * T_2) = z(T_1)z(T_2)$. Calculer $T_1 * T_2$ en fonction de T_1 et T_2. Que peut-on dire d'un argument de $z(T_1 * T_2)$? Que peut-on dire d'un argument de $z(T_1 * \widehat{T_1})$?

4. Si T_1 et T_2 sont réduits, en est-il de même de $T_1 * T_2$? Si T_1 et T_2 sont entiers, en est-il de même de $T_1 * T_2$? Si T_1 et T_2 sont primitifs, en est-il de même de $T_1 * T_2$?

5. Comparer les trios $T_1 * T_2$ et $T_2 * T_1$, $(T_1 * T_2) * T_3$ et $T_1 * (T_2 * T_3)$, T_1 et $T_1 * (1, 0, 0)$.

6. Étant donnés les trios T_1 et T_2, résoudre l'équation $T_1 * T = T_2$ où le trio T est l'inconnue.

7. Étant-donné un trio T, on définit une suite de trios (T_n) par $T_0 = (1, 0, 0)$ et $T_{n+1} = T * T_n$. Calculer $S(T_n)$. Étant donné un entier p, résoudre l'équation $T_1 * T = T_2$ où le trio T est l'inconnue.

Quatrième partie

On note A l'ensemble des entiers m non nuls tels qu'il existe deux entiers u, v tels que $m = u^2 + 3v^2$. On note A' l'ensemble des nombres complexes z non nuls tels qu'il existe deux entiers u, v tels que $z = u + iv\sqrt{3}$ (on remarquera que $|z|^2 = u^2 + 3v^2$). On note B l'ensemble des entiers n non nuls tels qu'il existe deux entiers r, s tels que $n = r^2 + rs + s^2$.

1. Montrer que le produit de deux éléments de A' appartient à A', puis que le produit de deux éléments de A appartient à A.

2. Montrer que, si p est un nombre premier élément de A, alors $p = 3$ ou 3 divise $p - 1$.

3. Montrer que $A = B$ (on pourra notamment remarquer que

$$r^2 + rs + s^2 = (r+s)^2 - (r+s)s + s^2)\,.$$

4. Montrer que 4 divise les éléments pairs de A et que les quotients appartiennent à A, puis que tout élément de A est produit d'un élément impair de A par une puissance de 4.

5. (a) Soit s'il en existe, un entier impair $m = u^2+3v^2$ tel que les entiers u et v soient premiers entre eux et qui admet un diviseur premier p n'appartenant pas à A. Montrer qu'il existe alors un plus petit entier strictement positif n_0 tel que $n_0 p$ appartienne à A. Montrer que n_0 est impair.

(b) Établir l'existence de deux entiers u' et v' inférieurs en valeur absolue à $p/2$ tels que p divise $u' - u$ et $v' - v$. Montrer que p divise l'entier non nul $u'^2 + 3v'^2$ et que $n_0 < p$.

(c) Établir l'existence de deux entiers non nuls premiers entre eux u_0 et v_0 tels que $n_0 p = u_0^2 + 3v_0^2$.

(d) Établir l'existence de deux entiers u_1 et v_1 inférieurs en valeur absolue à $n_0/2$ tels que n_0 divise $u_1 - u_0$ et $v_1 - v_0$. Montrer que n_0 divise l'entier non nul $u_1^2 + 3v_1^2$ que l'on notera $n_0 n_1$.

(e) En déduire qu'un tel entier m ne peut pas exister (on pourra considérer l'entier $n_0^2 n_1 p$).

6. Montrer que tout élément de A s'écrit $m = C^2 \cdot p_1 \cdots p_k$ où C est un entier naturel non nul et les p_i des nombres premiers distincts éléments de A.

7. (a) Soient p un nombre premier tel que 3 divise $p-1$, et K l'ensemble des triplets (x, y, z) où les entiers x, y et z sont strictement compris entre 0 et p, et tels que p divise $(xyz - 1)$. Montrer que K possède $(p-1)^2$ éléments, et que 3 divise le nombre d'éléments de K ne vérifiant pas $x = y = z$.

(b) En déduire qu'il existe un entier x strictement compris entre 1 et p tel que p divise $x^2 + x + 1$, puis que p appartient à A. Décrire les éléments de A.

8. Soit D l'ensemble des entiers d tels qu'il existe un trio entier (a, b, c) vérifiant $a + b + c = d$ et $abc \neq 0$. Montrer, grâce à la question 5 de la deuxième partie, que tout élément de D possède un diviseur premier élément de A. Réciproquement, que peut-on dire d'un entier non nul admettant un diviseur premier élément de A ?

9. En déduire les éléments de D compris au sens large entre 2001 et 2010.

Next we turn to solutions from our readers to problems of the 2000 Chinese Mathematical Olympiad given [2002 : 482–483].

1. In triangle ABC, $a \le b \le c$, where $a = BC$, $b = CA$ and $c = AB$, the circumradius is R and the inradius is r. What can be said about $\angle C$ if $a + b - 2R - 2r$ is

(a) positive? (b) zero? (c) negative?

Solved by Pierre Bornsztein, Maisons-Laffitte, France. We give the comment by Mohammed Aassila, Strasbourg, France.

This is problem 2690 [2001 : 534]. A solution appeared in [2002 : 549].

2. The sequence $\{a_n\}$ is defined by $a_1 = 0$, $a_2 = 1$ and

$$a_n \;=\; \frac{n}{2}a_{n-1} + \frac{n(n-1)}{2}a_{n-2} + (-1)^n\left(1 - \frac{n}{2}\right)$$

for $n \ge 3$. Simplify

$$a_n + 2\binom{n}{1}a_{n-1} + 3\binom{n}{2}a_{n-2} + \cdots + (n-1)\binom{n}{n-2}a_2 + n\binom{n}{n-1}a_1 \,.$$

Solution by Mohammed Aassila, Strasbourg, France.

First we show by induction that $a_n = (-1)^n + na_{n-1}$ for all $n \ge 2$. The relation is easily checked for $n = 2$. Consider any fixed integer $n \ge 3$. If $a_{n-1} = (-1)^{n-1} + (n-1)a_{n-2}$, then

$$\begin{aligned} a_n &= \tfrac{n}{2}a_{n-1} + \frac{n(n-1)}{2}a_{n-2} + (-1)^n\left(1 - \tfrac{n}{2}\right) \\ &= (-1)^n + \tfrac{1}{2}na_{n-1} + \tfrac{1}{2}n\big((-1)^{n-1} + (n-1)a_{n-2}\big) \\ &= (-1)^n + \tfrac{1}{2}na_{n-1} + \tfrac{1}{2}na_{n-1} \;=\; (-1)^n + na_{n-1}\,, \end{aligned}$$

as desired. Now, by induction again,

$$a_n \;=\; n! - \frac{n!}{1!} + \frac{n!}{2!} + \cdots + (-1)^n\frac{n!}{n!}\,,$$

which is the number of derangements of $(1, 2, \ldots, n)$.

5. Find all positive integers n for which there are k integers n_1, n_2, $\ldots$, n_k, each greater than 3, such that

$$n \;=\; n_1 n_2 \cdots n_k \;=\; \sqrt[2^k]{2^{(n_1-1)(n_2-1)\cdots(n_k-1)}} - 1\,.$$

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Aassila's write-up.

Let n be a positive integer, and let n_1, n_2, $\ldots$, n_k be integers, each greater than 3. Suppose that the equation in the problem is satisfied. Then

$$n \;=\; n_1 \cdots n_k \;=\; 2^m - 1\,, \qquad (1)$$

where

$$m = \frac{(n_1-1)(n_2-1)\cdots(n_k-1)}{2^k} = \left(\frac{n_1-1}{2}\right)\left(\frac{n_2-1}{2}\right)\cdots\left(\frac{n_k-1}{2}\right). \tag{2}$$

From (1), we see that m is a positive integer, since $2^m = n+1$ is a positive integer. We also see that n is odd and each n_i is odd. Since $n_i > 3$, we must have $n_i \geq 5$, for each i.

Lemma. For any integer $j \geq 10$, we have $2^j - 1 > j^3$.

Proof. We proceed by induction on j. The inequality is true for $j = 10$, since $2^{10} - 1 = 1023 > 10^3 = 1000$. Next, we assume that $2^j - 1 > j^3$ for some fixed integer $j \geq 10$. Let us prove that $2^{j+1} - 1 > (j+1)^3$. Since $j \geq 10$, we have

$$\left(\frac{j+1}{j}\right)^3 < \left(\frac{5}{4}\right)^3 < 2.$$

Thus, $2^{j+1} - 1 > 2^{j+1} - 2 = 2(2^j - 1) > 2j^3 > (j+1)^3$. ∎

Now, let us return to our problem. Suppose that $m \geq 10$. Using the lemma and (2), we have

$$2^m - 1 > m^3 = \left(\frac{n_1-1}{2}\right)^3 \cdots \left(\frac{n_k-1}{2}\right)^3. \tag{3}$$

For each i, since $n_i \geq 5$, we must have

$$\left(\frac{n_i-1}{2}\right)^3 \geq 4 \cdot \frac{n_i-1}{2} > n_i.$$

Using (3), we deduce that $2^m - 1 > n_1 n_2 \cdots n_k$, which contradicts (1).

Thus, there are no solutions with $m \geq 10$. It is easy to check that the only positive integer $m \leq 9$ for which (1) and (2) can be satisfied is $m = 3$, which gives $n = 7$.

Now we look at readers' solutions to problems of the 2000 Russian Mathematical Olympiad given in [2002 : 483–484].

1. Prove that there exist ten different real numbers a_1, a_2, …, a_{10} such that the equation

$$(x-a_1)(x-a_2)\cdots(x-a_{10}) = (x+a_1)(x+a_2)\cdots(x+a_{10})$$

has exactly 5 different real roots.

Solution by Mohammed Aassila, Strasbourg, France.

Let a_1, a_2, $\ldots$, a_{10} be distinct real numbers such that a_1, $\ldots$, a_5 are positive, $a_6 = 0$, $a_7 + a_8 = 0$, and $a_9 + a_{10} = 0$. For $k \in \{6, 7, 8, 9, 10\}$, the factor $x - a_k$ occurs on both sides of the given equation and, hence, a_k is a real root of the equation.

For $x \notin \{a_6, a_7, a_8, a_9, a_{10}\}$, the equation reduces to

$$(x - a_1)(x - a_2)\ldots(x - a_5) \;=\; (x + a_1)(x + a_2)\ldots(x + a_5)\,. \qquad (1)$$

If $x > 0$, then, for each $k \in \{1, 2, 3, 4, 5\}$,

$$|x - a_k| \;=\; \max\{x - a_k, a_k - x\} \;<\; x + a_k \;=\; |x + a_k|\,,$$

and (1) cannot hold. By symmetry, (1) cannot hold if $x < 0$. Neither does (1) hold if $x = 0$. Therefore, (1) has no real roots. Hence, the equation in the problem statement has no real roots besides a_6, a_7, a_8, a_9, a_{10}.

3. Let a_1, a_2, $\ldots$, a_{2000} be real numbers such that

$$a_1^3 + a_2^3 + \cdots + a_n^3 \;=\; (a_1 + a_2 + \cdots + a_n)^2$$

for all n, $1 \le n \le 2000$. Prove that every element of the sequence is an integer.

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bornsztein's solution.

First we note that if $a_i = i$ for $i = 1, 2, \ldots, n$, then

$$\begin{aligned} a_1^3 + a_2^3 + \cdots + a_n^3 \;&=\; \sum_{i=1}^{n} i^3 \;=\; \left(\frac{n(n+1)}{2}\right)^2 \\ &=\; \left(\sum_{i=1}^{n} i\right)^2 \;=\; (a_1 + a_2 + \cdots + a_n)^2\,. \end{aligned}$$

Lemma. If $a_i = i$ for $i = 1, 2, \ldots, n$ and $\sum\limits_{i=1}^{n+1} a_i^3 = \left(\sum\limits_{i=1}^{n+1} a_i\right)^2$, then $a_{n+1} \in \{n+1, -n, 0\}$.

Proof: Starting with the given relation $\sum\limits_{i=1}^{n+1} a_i^3 = \left(\sum\limits_{i=1}^{n+1} a_i\right)^2$, we get

$$\begin{aligned} a_{n+1}^3 + \sum_{i=1}^{n} i^3 \;&=\; \left(a_{n+1} + \sum_{i=1}^{n} i\right)^2 \\ &=\; a_{n+1}^2 + 2a_{n+1}\left(\frac{n(n+1)}{2}\right) + \left(\sum_{i=1}^{n} i\right)^2\,, \end{aligned}$$

which simplifies to $a_{n+1}(a_{n+1} + n)\big(a_{n+1} - (n+1)\big) = 0$. The conclusion of the lemma follows. ∎

For each positive integer k, let $\mathcal{P}_k$ be the claim:

If $a_1, a_2, \ldots, a_k$ are real numbers such that $\sum\limits_{i=1}^{n} a_i^3 = \left(\sum\limits_{i=1}^{n} a_i\right)^2$ for all integers n such that $1 \le n \le k$, then $a_1, a_2, \ldots, a_k$ are integers.

We will prove, by induction on k, that $\mathcal{P}_k$ holds for each k. First note that $\mathcal{P}_1$ holds, since the equation $a_1^3 = a_1^2$ implies that $a_1 \in \{0, 1\}$. Let $k \ge 1$ be a fixed integer, and suppose that $\mathcal{P}_i$ holds for $i = 1, 2, \ldots, k$. Let $a_1, a_2, \ldots, a_{k+1}$ be real numbers such that $\sum\limits_{i=1}^{n} a_i^3 = \left(\sum\limits_{i=1}^{n} a_i\right)^2$ for all integers n such that $1 \le n \le k+1$.

Case 1. There exists $i \in \{1, 2, \ldots, k+1\}$ such that $a_i = 0$.

Delete a_i from the sequence $a_1, a_2, \ldots, a_{k+1}$. The remaining k numbers in the sequence satisfy the hypothesis of $\mathcal{P}_k$. Since $\mathcal{P}_k$ holds, each of these numbers must be an integer. Thus, all of $a_1, a_2, \ldots, a_{k+1}$ are integers.

Case 2. There exists $i \in \{1, 2, \ldots, k+1\}$ such that $a_{i+1} = -a_i$.

Then $k \ge 2$. The fact that $\mathcal{P}_i$ holds implies that $a_1, a_2, \ldots, a_i$ are integers. Then $a_{i+1} = -a_i$ is an integer. Now delete a_i and a_{i+1} from the sequence $a_1, a_2, \ldots, a_{k+1}$. The remaining $k-1$ numbers in the sequence satisfy the hypothesis of $\mathcal{P}_{k-1}$. Since $\mathcal{P}_{k-1}$ holds, each of these numbers must be an integer. Thus, all of $a_1, a_2, \ldots, a_{k+1}$ are integers.

Case 3. For each $i \le k+1$, we have $a_i \ne 0$ and $a_i \ne -a_{i-1}$.

Then an easy induction, using the lemma above, shows that $a_i = i$ for all $i \in \{1, 2, \ldots, k+1\}$. Thus, all of $a_1, a_2, \ldots, a_{k+1}$ are integers.

This ends the induction step, and we are done.

5. Prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} < x\,,y \le 1\,.$$

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Aassila's write-up.

The problem should read: prove that

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} \le \frac{2}{\sqrt{1+xy}} \quad \text{for } 0 \le x,\, y \le 1.$$

If $x = 0$, then the inequality reduces to $1 + \dfrac{1}{\sqrt{1+y^2}} \le 2$, which is true, since $y > 0$. By symmetry, the inequality is also true if $y = 0$.

Now, suppose that $0 < x \le 1$ and $0 < y \le 1$. Let $u \ge 0$ and $v \ge 0$ be such that $x = e^{-u}$ and $y = e^{-v}$. Then the inequality becomes

$$\frac{1}{\sqrt{1+e^{-2u}}} + \frac{1}{\sqrt{1+e^{-2v}}} \le \frac{2}{\sqrt{1+e^{-(u+v)}}}\,;$$

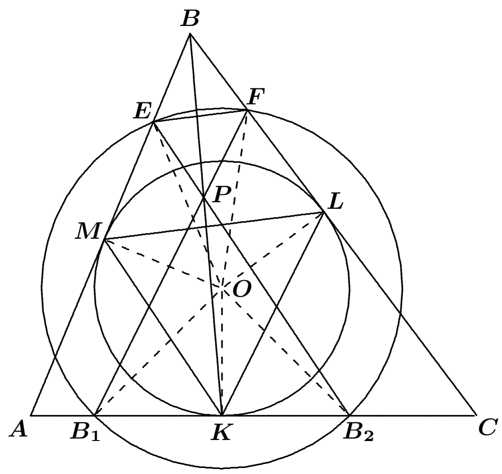
that is,

$$\frac{f(u)+f(v)}{2} \leq f\left(\frac{u+v}{2}\right),$$

where $f(x) = \dfrac{1}{\sqrt{1+e^{-2x}}}$. Since $f''(x) = \dfrac{1-2e^{2x}}{(1+e^{-2x})^{5/2}e^{4x}}$, the function f is concave on an interval containing $[0,\infty)$. Therefore, the above inequality is true.

6. The incircle of triangle ABC with centre O touches the side AC at K. Another circle with the same centre intersects each side at two points. The points of intersection on AC are B_1 and B_2, with B_1 closer to A. E is the point of intersection on AB closer to B, and F is the point of intersection on BC closer to B. Let P be the point of intersection of B_2E and B_1F. Prove that B, K, and P are collinear.

Solution by Toshio Seimiya, Kawasaki, Japan.



Let the incircle meet BC and BA at L and M, respectively. Let r be the radius of the incircle, and let r' be the radius of the other circle centred at O. Since $OK = OL = OM = r$ and $OB_1 = OB_2 = OE = OF = r'$, and since $\angle OKB_1 = \angle OKB_2 = \angle OLF = \angle OME = 90^\circ$, we have

$$\triangle OKB_1 \cong \triangle OKB_2 \cong \triangle OLF \cong \triangle OME.$$

Hence, $KB_1 = KB_2 = LF = ME$. Since $AK = AM$ and $KB_2 = ME$, we have $AK : KB_2 = AM : ME$. Thus, $MK \parallel EB_2$; that is, $MK \parallel EP$. Then, by symmetry, $LK \parallel FP$.

Similarly, since $BM = BL$ and $EM = FL$, we obtain $ML \parallel EF$. In triangles KLM and PFE, we have $KM \parallel PE$, $KL \parallel PF$, and $ML \parallel EF$. It follows that KP, LF, and ME are concurrent. Therefore, B, K, and P are collinear.

Now we turn to problems from the February 2003 number of the *Corner*. We give readers' solutions to the 2000 Korean Mathematical Olympiad, given in [2003 : 22–23].

1. Prove that for any prime p, there exist integers x, y, z, and w such that $x^2 + y^2 + z^2 - wp = 0$ and $0 < w < p$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give the version of Sinefakopoulos.

For $p = 2$, we set $x = y = w = 1$ and $z = 0$.

For p odd, we note that the numbers

$$1 + 0^2\,, \quad 1 + 1^2\,, \quad \dots\,, \quad 1 + \left(\frac{p-1}{2}\right)^2$$

leave $(p + 1)/2$ distinct remainders upon division by p, as do the numbers

$$-\,0^2\,, \quad -\,1^2\,, \quad \dots\,, \quad -\left(\frac{p-1}{2}\right)^2 .$$

Indeed, if $0 \le a,\, b \le (p - 1)/2$, then p divides $a^2 - b^2 = (a - b)(a + b)$ if and only if p divides $a + b$ or $a - b$, which is impossible if $a \ne b$.

Thus, there are at least two integers $0 \le x,\, y \le (p - 1)/2$ such that

$$1 + x^2 \;\equiv\; -y^2 \pmod{p} .$$

Hence, $x^2 + y^2 + 1 = wp$ for some integer w. Since

$$0 \;<\; x^2 + y^2 + 1 \;\le\; 2\left(\frac{p-1}{2}\right)^2 + 1 \;<\; p^2\,,$$

we have $0 < w < p$, as desired.

2. Determine all functions f from the set of real numbers to itself such that for every x and y,

$$f(x^2 - y^2) \;=\; (x - y)\big(f(x) + f(y)\big) .$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We feature Bornsztein's version.

The solution is the set of linear functions passing through the origin.

Let f be a function satisfying the given functional equation. Setting $x = y = 0$ in this equation leads to $f(0) = 0$. Then, setting $y = -x$ gives $0 = f(0) = 2x(f(x) + f(-x))$, from which we deduce that f is an odd function.

For all x and y, we have

$$f(x^2 - y^2) \;=\; (x-y)\big(f(x) + f(y)\big)$$

and also (replacing y by $-y$ and noting that $f(-y) = -f(y)$),

$$f(x^2 - y^2) \;=\; (x+y)\big(f(x) - f(y)\big)\,.$$

Hence,

$$(x-y)\big(f(x) + f(y)\big) \;=\; (x+y)\big(f(x) - f(y)\big)\,;$$

that is, $xf(y) = yf(x)$. Setting $y = 1$, we get $f(x) = xf(1)$. Thus, f is linear.

Conversely, it is easy to verify that any linear function passing through the origin is a solution of the problem.

4. Let p be a prime such that $p \equiv 1 \pmod 4$. Evaluate

$$\sum_{k=1}^{p-1}\left(\left\lfloor\frac{2k^2}{p}\right\rfloor\right) - 2\left(\left\lfloor\frac{k^2}{p}\right\rfloor\right)\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give the write-up of Aassila.

Let $\{x\}$ denote the fractional part of x; that is, $\{x\} = x - \lfloor x\rfloor$. Then

$$\begin{aligned}\left\lfloor\frac{2k^2}{p}\right\rfloor - 2\left\lfloor\frac{k^2}{p}\right\rfloor &= \left(\frac{2k^2}{p} - \left\{\frac{2k^2}{p}\right\}\right) - 2\left(\frac{k^2}{p} - \left\{\frac{k^2}{p}\right\}\right) \\ &= 2\left\{\frac{k^2}{p}\right\} - \left\{\frac{2k^2}{p}\right\}\,.\end{aligned}$$

If $\left\{\frac{k^2}{p}\right\} < \frac{1}{2}$, then

$$2\left\{\frac{k^2}{p}\right\} - \left\{\frac{2k^2}{p}\right\} \;=\; 2\left\{\frac{k^2}{p}\right\} - 2\left\{\frac{k^2}{p}\right\} \;=\; 0\,.$$

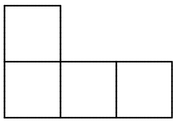
If $\left\{\frac{k^2}{p}\right\} \geq \frac{1}{2}$ then

$$2\left\{\frac{k^2}{p}\right\} - \left\{\frac{2k^2}{p}\right\} \;=\; 2\left\{\frac{k^2}{p}\right\} - \left(2\left\{\frac{k^2}{p}\right\} - 1\right) \;=\; 1\,.$$

Hence, $\sum\limits_{k=1}^{p-1}\left(\left\lfloor\frac{2k^2}{p}\right\rfloor - 2\left\lfloor\frac{k^2}{p}\right\rfloor\right)$ is equal to the number of $k \in [1, p-1]$ such that $\left\{\frac{k^2}{p}\right\} \geq \frac{1}{2}$; that is, the number of non-zero residues k modulo p such that k^2 is congruent to some number in $\left[\frac{p+1}{2}, p-1\right]$. Thus,

$$\sum_{k=1}^{p-1}\left(\left\lfloor\frac{2k^2}{p}\right\rfloor - 2\left\lfloor\frac{k^2}{p}\right\rfloor\right) = \frac{p-1}{2}\,.$$

5. Prove that an $m \times n$ rectangle can be constructed using copies of the following shape if and only if mn is a multiple of 8.



Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Tracy Walker, student, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Walker and Wang.

To make the conclusion correct, we must add the condition that $m > 1$ and $n > 1$.

Call a figure of the given shape a *tetramino*. We will prove that an $m \times n$ board admits a *perfect cover* by tetraminos if and only if $m > 1$, $n > 1$, and $8 \mid mn$.

We first show necessity. Suppose an $m \times n$ board has been covered by exactly t tetraminos. Then $mn = 4t$, which implies that $4 \mid mn$. Suppose, to the contrary, that $8 \nmid mn$. Then t is odd.

We now colour the rows of the board in an alternating manner so that all the odd numbered rows are coloured white and all the even numbered rows are coloured black. Since at least one of m and n is even, the total number of (unit) white squares must be even (and the total number of black squares is also even). It is also clear that a tetramino, regardless of its position on the board, must contain either three white squares and one black square (type I) or three black squares and one white square (type II). Let t_1 and t_2 denote the number of tetraminos of types I and II, respectively. Then $t_1 + t_2 = t$. Since t is odd, we deduce that t_1 and t_2 have opposite parity. The total number of white squares is $3t_1 + t_2$, which is odd. We have a contradiction.

We now establish the sufficiency. First note that a 2×4 board and a 3×8 board both admit perfect covers by tetraminos, as shown by Figure 1 and Figure 2, respectively:

1	2	2	2
1	1	1	2

Figure 1

1	1	3	3	3	4	6	6
1	2	3	4	4	4	5	6
1	2	2	2	5	5	5	6

Figure 2

Now consider an $m \times n$ board where $m > 1$, $n > 1$, and $8 \mid mn$. By considering the *transpose* of the board, if necessary, we may assume that either (i) $2 \mid m$ and $4 \mid n$ or (ii) $8 \mid n$. In case (i), a perfect cover may be obtained using $\frac{mn}{8}$ copies of Figure 1.

For case (ii), we invoke a well-known result of Frobenius which states that, if a, $b \in \mathbb{N}$ such that $(a,b) = 1$, then any natural number $m \geq (a-1)(b-1)$ can be written as $m = ra + sb$ where r, $s \in \mathbb{N} \cup \{0\}$. In particular, if $a = 2$ and $b = 3$, then any natural number m, except 1, can be written as $m = 2r + 3s$ for some non-negative integers r and s. Then it is clear that $\frac{nr}{4}$ copies of Figure 1 together with $\frac{ns}{8}$ copies of Figure 2 would provide a perfect cover.

6. The real numbers a, b, c, x, y, and z are such that $a > b > c > 0$ and $x > y > z > 0$. Prove that

$$\frac{a^2x^2}{(by+cz)(bz+cy)} + \frac{b^2y^2}{(cz+ax)(cx+az)} + \frac{c^2z^2}{(ax+by)(ay+bx)} \geq \frac{3}{4}.$$

Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give Bornsztein's solution.

We will prove the given inequality under the slightly weaker hypothesis that $a \geq b \geq c > 0$ and $x \geq y \geq z > 0$. For convenience, we let S denote the expression on the left side of the inequality.

From $a \geq b \geq c > 0$ and $x \geq y \geq z > 0$, we deduce that

$$a^2x^2 \geq b^2y^2 \geq c^2z^2. \tag{1}$$

Moreover,

$$0 < by + cz \leq cz + ax \leq ax + by$$

and

$$0 < bz + cy \leq cx + az \leq ay + bx.$$

Then

$$\begin{aligned}\frac{1}{(by+cz)(bz+cy)} &\geq \frac{1}{(cz+ax)(cx+az)} \\ &\geq \frac{1}{(ax+by)(ay+bx)} > 0.\end{aligned} \tag{2}$$

From (1), (2), and Chebyshev's Inequality, it follows that

$$\begin{aligned}S \geq\ & \tfrac{1}{3}(a^2x^2 + b^2y^2 + c^2z^2) \cdot \\ & \cdot \left(\frac{1}{(by+cz)(bz+cy)} + \frac{1}{(cz+ax)(cx+az)} + \frac{1}{(ax+by)(ay+bx)}\right).\end{aligned}$$

Using the AM–HM Inequality, we get

$$\frac{1}{3}\left(\frac{1}{(by+cz)(bz+cy)}+\frac{1}{(cz+ax)(cx+az)}+\frac{1}{(ax+by)(ay+bx)}\right)$$
$$\geq \frac{3}{(by+cz)(bz+cy)+(cz+ax)(cx+az)+(ax+by)(ay+bx)}\,.$$

Hence,

$$S \geq \frac{3(a^2x^2+b^2y^2+c^2z^2)}{S'}\,, \qquad (3)$$

where

$$\begin{aligned} S' &= (by+cz)(bz+cy)+(cz+ax)(cx+az)+(ax+by)(ay+bx) \\ &= a^2(xy+xz)+b^2(yz+yx)+c^2(zx+zy) \\ &\qquad +(ab+ac)x^2+(bc+ba)y^2+(ca+cb)z^2\,. \end{aligned}$$

Note that $xy+xz \leq \frac{1}{2}(x^2+y^2)+\frac{1}{2}(x^2+z^2) = x^2+\frac{1}{2}(y^2+z^2)$, with equality if and only if $x=y=z$. Using this and similar inequalities, we deduce that

$$\begin{aligned} S' &\leq a^2\left(x^2+\tfrac{1}{2}y^2+\tfrac{1}{2}z^2\right)+b^2\left(\tfrac{1}{2}x^2+y^2+\tfrac{1}{2}z^2\right) \\ &\qquad +c^2\left(\tfrac{1}{2}x^2+\tfrac{1}{2}y^2+z^2\right)+\left(a^2+\tfrac{1}{2}b^2+\tfrac{1}{2}c^2\right)x^2 \\ &\qquad\quad +\left(\tfrac{1}{2}a^2+b^2+\tfrac{1}{2}c^2\right)y^2+\left(\tfrac{1}{2}a^2+\tfrac{1}{2}b^2+c^2\right)z^2 \\ &= (a^2+b^2+c^2)(x^2+y^2+z^2)+a^2x^2+b^2y^2+c^2z^2\,, \end{aligned}$$

with equality if and only if $x=y=z$ and $a=b=c$. Using Chebyshev's Inequality again, we get

$$\begin{aligned} S' &\leq 3(a^2x^2+b^2y^2+c^2z^2)+a^2x^2+b^2y^2+c^2z^2 \\ &= 4(a^2x^2+b^2y^2+c^2z^2)\,. \qquad (4) \end{aligned}$$

From (3) and (4), we deduce that $S \geq \frac{3}{4}$. Equality occurs if and only if $x=y=z$ and $a=b=c$.

We conclude this number of the *Corner* with the only solution we have on file to problems of the 2000 Vietnamese Mathematical Olympiad given [2003 : 24–25]. Here is an opportunity for readers to rise to the challenge of filling in the gaps!

4. For every integer $n \geq 3$ and any given angle α in $(0,\pi)$, let $P_n(x) = x^n\sin\alpha - x\sin n\alpha + \sin(n-1)\alpha$.

(a) Prove that there is only one polynomial of the form $f(x) = x^2+ax+b$ such that for every $n \geq 3$, $P_n(x)$ is divisible by $f(x)$.

(b) Prove that there does not exist a polynomial $g(x)$ of the form $g(x) = x+c$ such that for every $n \geq 3$, $P_n(x)$ is divisible by $g(x)$.

Solved by Michel Bataille, Rouen, France.

First we observe that

$$\begin{aligned} P_{n+1}(x) - xP_n(x) & \\ &= x^2 \sin n\alpha - x\big(\sin(n+1)\alpha + \sin(n-1)\alpha\big) + \sin n\alpha \\ &= x^2 \sin n\alpha - 2x \sin n\alpha \cos\alpha + \sin n\alpha \\ &= (\sin n\alpha)(x^2 - 2x\cos\alpha + 1)\,. \end{aligned}$$

(a) Let $f(x) = x^2 + ax + b$, and suppose that $f(x)$ divides $P_n(x)$ for all $n \geq 3$. Then $f(x)$ divides $P_{n+1}(x) - P_n(x) = (\sin n\alpha)(x^2 - 2x\cos\alpha + 1)$ for all $n \geq 3$. Choosing n such that $\sin n\alpha \neq 0$ ($n = 3$ or $n = 4$ will do), we see that necessarily $f(x) = x^2 - 2x\cos\alpha + 1$.

Conversely, since

$$\begin{array}{rcccl} P_n(x) & = & xP_{n-1}(x) & + & \big(\sin(n-1)\alpha\big)f(x) \\ P_{n-1}(x) & = & xP_{n-2}(x) & + & \big(\sin(n-2)\alpha\big)f(x) \\ \vdots & & \vdots & & \vdots \\ P_2(x) & = & 0 & + & (\sin\alpha)f(x)\,, \end{array}$$

we have $P_n(x) = f(x)\big(\sin(n-1)\alpha + x\sin(n-2)\alpha + \cdots + x^{n-2}\sin\alpha\big)$. Thus, $f(x)$ divides every $P_n(x)$ with $n \geq 3$.

(b) Suppose $g(x) = x + c$ divides $P_n(x)$ for all $n \geq 3$. In particular, $g(x)$ divides $P_3(x)$. From (a), we have $P_3(x) = (\sin\alpha)f(x)(x + 2\cos\alpha)$. If we assume that c is a real number, then $g(x)$ does not divide $f(x)$, since $f(x)$ has two non-real roots, $e^{i\alpha}$ and $e^{-i\alpha}$. Therefore, $g(x)$ must divide $x + 2\cos\alpha$, and we must have $c = -2\cos\alpha$.

Since $g(x)$ divides $P_4(x)$, it follows that $P_4(-2\cos\alpha) = 0$, which yields $3 - 4\sin^2\alpha = 0$. Then $\alpha = \frac{\pi}{3}$ or $\alpha = \frac{2\pi}{3}$. But in each of these cases, $-2\cos\alpha$ is not a root of $P_5(x)$. Thus, $g(x) = x + c$ cannot divide all $P_n(x)$ (if c is real).

That completes the *Corner* for this issue. Send me your Olympiad contest materials and your nice solutions and generalizations to problems in the *Corner*.