SKOLIAD No. 84

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Please send your solutions to the problems in this edition by ***1 June, 2005***. A copy of **MATHEMATICAL MAYHEM Vol. 2** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (*) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

Our items this issue come from a contest organized by the South african Mathematical Society and sponsored by the Actuarial Society of South Africa.

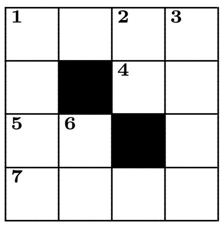
South African Interprovincial Mathematics Olympiad 2004

Team Paper, Juniors: 60 minutes allowed

1. (*) Five bags of rice are weighed two at a time, in all possible combinations. The ten weights are 72, 73, 76, 77, 79, 80, 81, 83, 84, and 87. What are the weights of the five bags?

2. (*) Delete 60 digits from the number 1 2 3 4 5 6 . . . 38 39 40 in such a way as to make the resulting number as small as possible.

3. Solve the crossnumber puzzle:

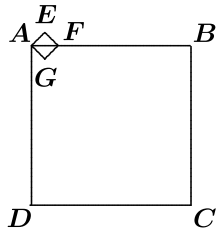


Across	**Down**
1. Cube of a prime	1. Square of a prime
4. Square	2. Three times cube root of 1 Across
5. Square	3. Square of a prime
7. Cube	6. Twice cube root of 7 Across

4. Find the sum of the digits of $10^{2004} - 2004$.

5. An urn contains 100 balls of different colours: namely, 10 white, 10 black, 12 yellow, 14 blue, 24 green, and 30 red. What is the minimum number of balls that must be drawn from the urn without looking if you want to be certain that at least 15 of the balls drawn are of the same colour?

6. In the diagram at right, square $ABCD$ has side 24 cm and square $AEFG$ has side 2 cm. What is the length of CE in cm?



7. All the positive integers, starting with 1, are written in order, namely,

$$12345678910111213141516\ldots$$

Find the digit appearing in the 206 788$^{\text{th}}$ position.

8. How many times does the number 2 appear when the product

$$1002 \cdot 1003 \cdot 1004 \cdots 2004$$

is expanded into its prime factors?

9. In the addition below, digits have been replaced by letters in a one-to-one fashion. Given that $D = 5$, work out the original numbers.

D	O	N	A	L	D
G	E	R	A	L	D
R	O	B	E	R	T

10. Consider a square having 16 cells each containing a plus sign or a minus sign. Suppose we change all the signs in a given row (or column), doing this several times until the number of minus signs is a minimum. A table that has the property that any such change does not decrease the number of minus signs is called a *minimal table*, and the number of minus signs in a minimal table is called the *characteristic* of the table. Find all possible values of the characteristic.

Olympiade Mathématique Interprovinciale d'Afrique du Sud 2004

Épreuve en Équipe, Junior : 60 minutes permises

1. ($*$) Cinq sacs de riz sont pesés deux à la fois dans toutes les combinaisons. Les dix pesées obtenues sont 72, 73, 76, 77, 79, 80, 81, 83, 84 et 87. Combien pèsent chacun des sacs ?

2. ($*$) Enlever 60 chiffres du nombre 1 2 3 4 5 6 ... 38 39 40 pour que le nombre résultant soit le plus petit possible.

3. Résoudre le nombre croisé :

1		2	3
	■	4	
5	6	■	
7			

Horizontal

1. Cube d'un premier
4. Carré
5. Carré
7. Cube

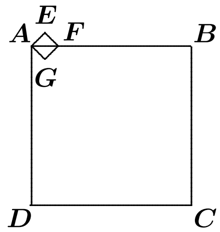
Vertical

1. Carré d'un premier
2. Trois fois la racine cubique de 1 Horizontal
3. Carré d'un premier
6. Double de la racine cubique de 7 Horizontal

4. Trouver la somme des chiffres de $10^{2004} - 2004$.

5. Une urne contient 100 balles de couleurs différentes, soit 10 blanches, 10 noires, 12 jaunes, 14 bleues, 24 vertes et 30 rouges. Quelle est le nombre minimal de balles pigées à l'aveugle que l'on doit enlever pour s'assurer qu'il y a 15 balles de la même couleur parmi les balles tirées.

6. Dans le dessin à droite, le carré $ABCD$ a 24 cm de côté alors que $AEFG$ en a 2 cm. Que mesure CE en cm ?



7. Tous les entiers positifs à partir de 1 sont écrits en ordre, soit :

$$12345678910111213141516\ldots$$

Quel est le 206 788$^{\text{ieme}}$ chiffre ?

8. Combien de fois le nombre 2 apparaît quand

$$1002 \cdot 1003 \cdot 1004 \cdots 2004$$

est décomposé en facteurs premiers ?

9. Dans l'addition suivante, les chiffres ont été remplacés par des lettres une à une. Sachant que $D = 5$, trouver les nombres originaux.

$$\begin{array}{cccccc} D & O & N & A & L & D \\ G & E & R & A & L & D \\ \hline R & O & B & E & R & T \end{array}$$

10. Considérer un carré ayant 16 cases contenant chacune des plus et des moins. Supposons que l'on change tous les signes dans une rangée (ou une colonne) à répétition jusqu'à ce que le nombre de moins soit minimal. Une table qui a la propriété de ne pas pouvoir se faire réduire le nombre de moins est appelée *minimale* et le nombre de moins s'y trouvant est la *caractéristique* de la table. Trouver toutes les caractéristiques possibles.

Next we give the solutions to the Nova Scotia Math League championship [2004 : 321–322].

Nova Scotia Math League Game 4: Group Questions

1. A *lattice point* is a point (x, y), where the coordinates are both integers. For example, $(3, -4)$ and $(5, 0)$ are lattice points, but $(2, 4.58)$ is not.

Determine the number of lattice points on the circumference of the circle $x^2 + y^2 = 25$.

Solution by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON, modified by the editor.

If a lattice point has coordinates (m, n) and lies on the circumference of the circle $x^2 + y^2 = 25$, then $m^2 + n^2 = 25$. Since m and n are integers and $m^2 \geq 0$, it follows that n^2 is in $\{0, 1, 4, 9, 16, 25\}$. By symmetry, this also holds for m^2.

- If $n^2 = 1$ or $n^2 = 4$, then m^2 is not a perfect square ($m^2 = 24$ or $m^2 = 21$) and yields no lattice points on the circle.

- If $n^2 = 0$, then $n = 0$ and $m^2 = 25$, which yields $m = \pm 5$. Thus, we get the two lattice points $(-5, 0)$ and $(5, 0)$.

- If $n^2 = 9$, then $n = \pm 3$ and $m^2 = 25 - 9 = 16$, which yields $m = \pm 4$. Hence, we get the four lattice points $(-4, -3)$, $(-4, 3)$, $(4, -3)$, and $(4, 3)$.

- If $n^2 = 16$, then $n = \pm 4$ and $m^2 = 25 - 16 = 9$, which yields $m = \pm 3$. Thus, we get the four lattice points $(-3, -4)$, $(-3, 4)$, $(3, -4)$, and $(3, 4)$.

- Finally, if $n^2 = 25$, then $n = \pm 5$ and $m^2 = 0$, which means that $m = 0$. This gives the two lattice points $(0, -5)$ and $(0, 5)$.

Therefore, there are 12 lattice points in total on the circle.

One incorrect solution was received.

2. Let $S = \frac{2^2-1}{2^2} \times \frac{3^2-1}{3^2} \times \frac{4^2-1}{4^2} \times \cdots \times \frac{2004^2-1}{2004^2}$. Express S as a fraction, reduced to lowest terms.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

$$\begin{aligned} S &= \frac{\prod_{k=2}^{2004}(k+1)\prod_{k=2}^{2004}(k-1)}{\prod_{k=2}^{2004} k \prod_{k=2}^{2004} k} = \frac{\prod_{k=3}^{2005} k \prod_{k=1}^{2003} k}{\prod_{k=2}^{2004} k \prod_{k=2}^{2004} k} \\ &= \frac{2005}{2 \cdot 2004} = \frac{2005}{4008}. \end{aligned}$$

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

3. A *palindrome* is a number that reads the same forwards and backwards, such as 8338 and 50705.

Let A and B be four-digit palindromes, and let C be a five-digit palindrome. If $A + B = C$, determine all possible values of C.

Solution by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON, modified by the editor.

Let the palindrome A be $abba$, where a and b are digits with $a \neq 0$. Similarly, let the palindrome B be $cddc$, where c and d are digits with $c \neq 0$. Since A and B are four-digit numbers, their sum, $A + B = C$, is at most 19998. Therefore, the left-most digit of C must be 1. Let the palindrome C be $1fgf1$, where f and g are digits.

Since the units digit of C is 1, we must have $a + c = 1$ or $a + c = 11$. But only $a + c = 11$ is possible, because $a \neq 0$ and $c \neq 0$. This means that $f = 1$ or $f = 2$ (look at the thousands column of the addition).

The addition now looks like

$$\begin{array}{ccccc} & & & 1 & \\ & a & b & b & a \\ + & c & d & d & c \\ \hline 1 & f & g & f & 1 \end{array}$$

where $f = 1$ or $f = 2$ and the first row represents "carries".

Case 1: $f = 1$.

Because $a + c = 11$ and $f = 1$, we see that $b + d$ ends with 0 (look at the tens column). If $b+d = 10$ and $a+c = 11$, then, looking at the left-most f in C, we see that f must be 2, a contradiction. Hence, $b + d = 0$, from which we have $g = 0$, which gives us $C = 11011$.

Case 2: $f = 2$.

Since $a + c = 11$ and $f = 2$, then $b + d$ ends with 1. Thus, $b + d = 1$ or $b + d = 11$. If $b + d = 1$ and $a + c = 11$, then, looking at the left-most f in C, we see that f has to be 1, a contradiction. Hence, $b + d = 11$, from which we have $g = 2$, which gives us $C = 12221$.

Therefore, the possible palindromes are 11011 and 12221.

One incomplete solution was received.

4. The circle with equation $x^2 + y^2 = 1$ intersects the line $y = 7x + 5$ at two distinct points, A and B. Let O be the centre of the circle. Find the measure of $\angle AOB$.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

It is easy to find A and B as $\left(-\frac{3}{5}, \frac{4}{5}\right)$ and $\left(-\frac{4}{5}, -\frac{3}{5}\right)$, respectively. The distance between these points is $\sqrt{2}$, and each of them is a distance 1 from O. Hence, AOB is a triangle with sides $AO = BO = 1$ and $AB = \sqrt{2}$. By the converse of the Pythagorean Theorem, $\angle AOB = \pi/2$.

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

5. Determine all integers x such that $(x^2 - 3x + 1)^{x+1} = 1$.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON, modified by the editor.

The given equation has the form $b^y = 1$, where $b = x^2 - 3x + 1$ and $y = x + 1$. The equation $b^y = 1$ holds for integers b and y in the following cases:

(a) $b = 1$. Then $x^2 - 3x = 0$; whence, $x = 0$ or $x = 3$.

(b) $b = -1$ and y is even. Then $(x-2)(x-1) = 0$ and $x+1$ even; whence, $x = 1$.

(c) $y = 0$ and $b \neq 0$. Then $x = -1$.

Therefore, the solutions are $x = -1$, $x = 0$, $x = 1$, and $x = 3$.

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

6. Let $f(a, b)$ denote the sum of the integers between a and b, inclusive. For example, $f(1, 5) = 1+2+3+4+5 = 15$ and $f(3, 6) = 3+4+5+6 = 18$.

Determine the value of $f(133333, 533333)$.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Let $a = 133333$. Then we have the arithmetic series

$$a + (a + 1) + \cdots + (a + 400000) .$$

The sum of this series is

$$\begin{aligned}400001a + 200000(400001) &= (400001)(333333)\\ &= 3(44444511111)\\ &= 133333533333\,.\end{aligned}$$

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

7. A hexagon and an equilateral triangle have equal perimeters. If the area of the hexagon is $6\sqrt{3}$ square units, what is the area of the triangle?

Solution by the editors.

The problem intended the hexagon to be regular. Let a be the length of a side of the hexagon. Then the equilateral triangle has side-length $2a$. When we subdivide the hexagon into 6 small equilateral triangles of side-length a we see that each of them must have area $\sqrt{3}$. Since the equilateral triangle of side-length $2a$ has twice the base and twice the altitude of the smaller triangle, it must have 4 times the area. Thus, the area of the original triangle is $4\sqrt{3}$.

Two incorrect solutions were received.

8. Determine all values of x for which

$$(1999x - 99)^3 = (1234x - 56)^3 + (765x - 43)^3\,.$$

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON, modified by the editor.

Factoring the sum of cubes on the right side of the equation, we find that one factor is $(1999x - 99)$; thus, one solution of the equation is $x = 99/1999$.

Now we rewrite the equation as

$$(1999x - 99)^3 - (1234x - 56)^3 = (765x - 43)^3\,.$$

Factoring the difference of cubes on the left side, we find that one factor is $(765x - 43)$; thus, one solution is $x = 43/765$.

Finally, we rewrite the equation as

$$(1999x - 99)^3 - (765x - 43)^3 = (1234x - 56)^3\,.$$

Factoring again, we see that $x = 56/1234$ is a solution.

Since the original equation is cubic in x, it has only 3 roots, and we have them all.

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

9. Find all solutions (x, y) in real numbers to

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{5}{6}, \\ x^2y + xy^2 &= 30. \end{aligned}$$

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Let $p = x + y$ and $q = xy$. The first equation gives us $6p = 5q$. The second equation gives us $pq = 30$. Solving, we first obtain $p = 5q/6$ and then $q^2 = 36$. Therefore, we have $q = \pm 6$. If $q = 6$, then $p = 5$; if $q = -6$, then $p = -5$. Solving for x and y, we get the solutions $(x, y) = (3, 2)$, $(2, 3)$, $(-6, 1)$, and $(1, -6)$.

Also solved by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

10. Find the number of real solutions to the equation $\sin(x) = x/315$.

Solution by Alex Remorov, grade 8 student, Waterloo Collegiate Institute, Kitchener-Waterloo, ON.

If $x = 0$, we have $\sin x = 0 = x/315$, which means that $(0, 0)$ is a solution.

The number of solutions is the number of points of intersection of the curve $y = \sin x$ with the line $y = x/315$. We know that $\sin x$ is periodic with period 2π and that $-1 \leq \sin x \leq 1$. Hence, if the line $y = x/315$ is to intersect $y = \sin x$, we need to have $-1 \leq x/315 \leq 1$; that is, $-315 \leq x \leq 315$. Since $315 \approx 100.27\pi$, we can write these inequalities as $-100.27\pi \leq x \leq 100.27\pi$.

There is no solution to the equation for $-100.27\pi \leq x \leq -100\pi$, since we then have $-1 \leq x/315 \leq -0.997$ and $-0.75 \leq \sin x \leq 0$. The same goes for $100\pi \leq x \leq 100.27\pi$. We can also see that there will be 2 solutions in each of the 100 intervals $-100\pi \leq x \leq -98\pi$, $-98\pi \leq x \leq -96\pi$, ..., $-2\pi \leq x \leq 0$, $0 \leq x \leq 2\pi$, ..., $98\pi \leq x \leq 100\pi$. But this counts the solution $x = 0$ twice; thus, we have $2 \times 100 - 1 = 199$ solutions.

Also solved by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

That brings us to the end of another Skoliad. Continue to send in your contests and solutions.