

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of CARL LIBIS, University of Rhode Island, Kingston, RI, USA from the list of solvers of 2862(a).

2771★. [2002 : 399; 2003 : 405] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Find all pairs of positive integers a and b such that

$$(a + b)^b = a^b + b^a.$$

A new solution by Manuel Benito, Óscar Ciaurri, and Emilio Fernández, Logroño, Spain. Their earlier solution [2003 : 406–407] made use of the Fermat-Wiles Theorem. The solution below uses only elementary number theory. The initial part is essentially the same as before.

Clearly, $(a, 1)$ is a solution for all positive integers a . We show that these are the only solution pairs.

Assuming that the equation holds for $b > 1$, we have

$$a^b + b^a = (a + b)^b = \sum_{k=0}^b \binom{b}{k} a^{b-k} b^k > a^b + b^b,$$

and thus, $a > b > 1$. Let $d = \gcd(a, b)$. Set $a_1 = a/d$ and $b_1 = b/d$. Then $a_1 > b_1$ and $\gcd(a_1, b_1) = 1$, and the given equation becomes

$$(a_1 + b_1)^b = a_1^b + d^{a-b} b_1^a, \quad (1)$$

or equivalently (applying the Binomial Theorem and dividing by b_1^2),

$$d a_1^{b-1} + \binom{b}{2} a_1^{b-2} + \cdots + \binom{b}{b-1} a_1 b_1^{b-3} + b_1^{b-2} = d^{a-b} b_1^{a-2}. \quad (2)$$

First, suppose that $b_1 = 1$. Then $d = b > 1$ and $a_1 > b_1 = 1$, and equation (1) becomes

$$(a_1 + 1)^d = a_1^d + d^{d(a_1-1)}. \quad (3)$$

- (i) If $a_1 = 2$, then equation (3) becomes $3^d = 2^d + d^d$. But this is not possible, since $3^d > 2^d + d^d$ when $d = 1$ or $d = 2$, and $3^d < 2^d + d^d$ when $d \geq 3$.
- (ii) If $a_1 = 3$, then equation (3) becomes $4^d = 3^d + (d^2)^d$. But this is impossible, since $4^d < 3^d + (d^2)^d$ for all $d > 1$.

- (iii) If $a_1 > 3$, then equation (3) cannot hold because $a_1 + 1 < d^{a_1-1}$ for all $d > 1$.

We conclude that $b_1 \neq 1$.

Next suppose that $b_1 = 2$. Then $b = 2d$. Furthermore, a_1 is odd (since $\gcd(a_1, b_1) = 1$), and equation (1) can be rewritten as

$$(a_1 + 2)^{2d} = a_1^{2d} + 2^{a_1 d} d^{d(a_1-2)}. \quad (4)$$

- (i) If $a_1 = 3$, then equation (4) becomes $25^d = 9^d + 8^d d^d$. We can easily check that this equation does not hold for $d = 1$, $d = 2$, or $d = 3$. It does not hold for $d > 3$, since $25 < 8d$ when $d > 3$.
- (ii) If $a_1 = 5$, then equation (4) becomes $49^d = 25^d + 32^d d^{3d}$. But, in fact, $49^d < 25^d + 32^d d^{3d}$ for all $d \geq 1$.
- (iii) If $a_1 > 6$, then equation (4) cannot hold because $(a_1 + 2)^2 < 2^{a_1}$.

We conclude that $b_1 \neq 2$.

Now suppose that $b_1 = 2^\ell$ with $\ell > 1$. Then $a_1 > 2^\ell$, and equation (2) becomes

$$da_1^{b-1} + 2^{\ell-1} d(2^\ell d - 1) a_1^{b-2} + \binom{b}{3} 2^\ell a_1^{b-3} + \dots + 2^{\ell(b-2)} = 2^{\ell(a-2)} d^{a-b}.$$

The first term on the left side of this equation is $da_1^{b-1} = da_1^{2^\ell d-1}$. This term must be even, since the other terms in the equation all contain factors of 2. But a_1 is odd, since b_1 is even and $\gcd(a_1, b_1) = 1$. Therefore, $a_1^{2^\ell d-1}$ is odd. This means that d is even, say $d = 2d_1$. Now equation (1) becomes

$$\left((a_1 + 2^\ell)^{2^{\ell-1} d_1}\right)^4 = \left(a_1^{2^{\ell-1} d_1}\right)^4 + \left(2^{\ell a_1 d_1} d_1^{d_1(a_1-2^\ell)}\right)^2.$$

But this equation cannot hold because the equation $x^4 - y^4 = z^2$ has no non-trivial integer solutions, as is well known.

It only remains for us to rule out the case where b_1 is a multiple of some odd prime p . In this case, we must have $a_1 > b_1 \geq 3$ and $\gcd(a_1, p) = 1$ (since $\gcd(a_1, b_1) = 1$). Since p divides b_1 , it follows from equation (2) that p divides da_1^{b-1} . Since a_1 and p are relatively prime, we conclude that d is a multiple of p . Let $d = p^k d_1$ and $b_1 = p^\ell b_2$, where d_1 and b_2 are each relatively prime to p , and k and ℓ are positive integers. Then $b = p^{k+\ell} d_1 b_2$, and equation (2) becomes

$$p^k d_1 a_1^{b-1} + \binom{b}{2} a_1^{b-2} + \binom{b}{3} a_1^{b-3} p^\ell b_2 + \dots + \binom{b}{b-1} a_1 p^{\ell(b-3)} b_2^{b-3} + p^{\ell(b-2)} b_2^{b-2} = p^{\ell(a-2)+k(a-b)} b_2^{a-2} d_1^{a-b}. \quad (5)$$

In the above equation, each term $t_m = \binom{b}{m} a_1^{b-m} p^{\ell(m-2)} b_2^{m-2}$ for $m \geq 2$ is divisible by the prime power p^{k+1} . To see this, there are two cases to consider.

Case (i): $p^j < m < p^{j+1}$ for some non-negative integer j .

Then

$$\binom{b}{m} = p^{k+\ell} d_1 b_2 \cdot \frac{p^{k+\ell} d_1 b_2 - 1}{1} \cdots \frac{p^{k+\ell} d_1 b_2 - p^j}{p^j} \cdots \frac{p^{k+\ell} d_1 b_2 - m + 1}{m - 1} \cdot \frac{1}{m},$$

which is a multiple of $p^{k+\ell}$ and therefore a multiple of p^{k+1} . It follows that p^{k+1} divides t_m .

Case (ii): $m = p^j$ for some positive integer j .

Then

$$\binom{b}{m} = p^{k+\ell} d_1 b_2 \cdot \frac{p^{k+\ell} d_1 b_2 - 1}{1} \cdots \frac{p^{k+\ell} d_1 b_2 - p^j + 1}{p^j - 1} \cdot \frac{1}{p^j},$$

which is a multiple of $p^{k+\ell-j}$. From this we see that t_m is divisible by

$$p^{k+\ell-j} p^{\ell(p^j-2)} = p^{k+\ell-j+p^j\ell-2\ell} = p^{k+(p^j-1)\ell-j}.$$

It follows that t_m is divisible by p^{k+1} (since $(p^j - 1)\ell - j \geq 1$).

Thus, in both cases, we see that t_m is divisible by p^{k+1} , as claimed.

From (5) we now see that $d_1 a_1^{b-1}$ is a multiple of p , which contradicts the fact that a_1 and d_1 are each relatively prime to p .

Consequently, there are no solutions to the equation in the problem statement other than the pairs $(a, 1)$, where a is any positive integer.

2868. [2003 : 399; 2004 : 379] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$, we have $c^4 = a^4 + b^4$.

- (a) Show that $\triangle ABC$ is acute angled.
- (b) Determine the range of $\angle ACB$.
- (c)★ How can we generalize to $c^n = a^n + b^n$?

Ed: This problem was the subject of a recent article by Guanshen Ren in *The College Mathematics Journal* (Sept 2004, pp. 305–307). The readership may be interested to compare the approach taken by Ren with that of our featured solver, Chip Curtis.

2886. [2003 : 468; 2004 : 519] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

If a, b, c are positive real numbers such that $abc = 1$, prove that

$$ab^2 + bc^2 + ca^2 \geq ab + bc + ca.$$

Editor's Comment: Mea culpa! We goofed! Two of the three featured solutions to this problem, which appeared in [2004 : 519], are flawed.

Vedula N. Murty, Dover, PA, USA, and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON noticed that Solution II is flawed. Using the Cauchy–Schwarz Inequality on the vectors $\left[\sqrt{\frac{b}{c}}, \sqrt{\frac{c}{a}}, \sqrt{\frac{a}{b}}\right]$ and $\left[\frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{a}}\right]$ yields $\left(\frac{b}{c} + \frac{c}{a} + \frac{a}{b}\right)\left(\frac{1}{b} + \frac{1}{c} + \frac{1}{a}\right) \geq \left(\frac{1}{\sqrt{c}} + \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right)^2$, from which inequality (1) does not follow.

Yufei Zhao and D. Kipp Johnson, Beaverton, OR, USA indicated that Solution III is flawed, since Muirhead's Theorem does not yield inequality (2) as claimed. Muirhead's Theorem states that if x, y, z are positive and the vector $[p, q, r]$ majorizes the vector $[u, v, w]$, then

$$\sum_{\text{permutation}} x^p y^q z^r \geq \sum_{\text{permutation}} x^u y^v z^w,$$

where there are six permutations in the case of a 3–vector. For this problem, the above inequality becomes

$$\begin{aligned} & 2x^3 y^3 z^0 + 2x^3 y^0 z^3 + 2x^0 y^3 z^3 \\ & \geq x^3 y^2 z^1 + x^3 y^1 z^2 + x^2 y^3 z^1 + x^2 y^1 z^3 + x^1 y^3 z^2 + x^1 y^2 z^3, \end{aligned}$$

from which inequality (2) does not follow. However, since Solution III has an alternate method of establishing (2) via the AM–GM Inequality, it still represents a valid approach.

2897. [2003 : 518, 2004 : 530] *Proposed by Václav Konečný, Big Rapids, MI, USA.*

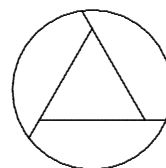
(a) Show that it is possible to divide a circular disc into four parts with the same area by means of three line segments of the same length.

(b) Does there exist a straight edge and compass construction (in the classical sense; that is, with a finite number of steps)?

Comment by Dave Ehren and Stan Wagon, Macalester College, St. Paul, MN, USA.

The featured solution to part (b) of this problem [2004 : 530] attempted to show that no arrangement of lines that satisfies part (a) can be constructed by straight edge and compass. The assertion in the opening statement of the argument is false, we believe, and hence the proof is incorrect.

For suppose the disc has area A . Place an equilateral triangle having area $A/4$ so that it is centred in the disc. This triangle will not touch the circle. Extend each side until it hits the circle, as in the figure. This construction partitions the disc into four parts with the same area, but it does not have a chord, which the proof of (b) says is necessary.



The circumradius of the equilateral triangle is $\sqrt{\pi/(3\sqrt{3})}$ if the original circle has radius 1. Hence, this particular partition of the disc cannot be constructed with straight edge and compass. But there is still a possibility that some other partition could be found.

Editor's Comments: Part (b) seems to have two legitimate interpretations. A narrow interpretation is that the construction used in part (a) cannot be made by ruler and compass. Except for the featured solution, all submitted solutions (including the proposer's) correctly proved this. The more general question—whether an arrangement of three equal segments that divide the disc into four parts of equal area can be constructed using Euclidean tools—remains open.

2914. [2004 : 106, 109] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

On the sides of an acute-angled triangle ABC , similar isosceles triangles DBC , ECA , FAB are constructed externally, such that

$$\begin{aligned}\angle DBC &= \angle DCB = \angle EAC = \angle ECA \\ &= \angle FAB = \angle FBA = \angle BAC.\end{aligned}$$

Let M be the mid-point of BC , and let P and Q be the intersections of DE with AC and of DF with AB , respectively.

Prove that $MP : MQ = AB : AC$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Locate N on the ray AC such that $AN = FB$. Then $AFBN$ is a rhombus. Since $AE : AN = AC : AB$, we see that $\triangle AEN \sim \triangle ACB$. Hence, $EN = CD$ and $EN \parallel CD$ [because of equal alternate angles with respect to the transversal CN]. Therefore, $ENDC$ is a parallelogram. Consequently, P is the mid-point of CN . Thus, $MP = \frac{1}{2}BN = \frac{1}{2}AF$. Similarly, $MQ = \frac{1}{2}AE$. Hence, $MP : MQ = AF : AE = AB : AC$.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Janous, who used rectangular coordinates, points out that his proof makes no use of the condition that $\triangle ABC$ is acute-angled. The featured argument clearly is valid were $\angle B$ or $\angle C$ not acute. This editor checked that it is valid also when $\angle A$ is obtuse, but the picture becomes cluttered with overlapping triangles. Of course $\angle A$ could not be a right angle, because then points D , E , F would not be defined.

2915. [2004 : 106, 109] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

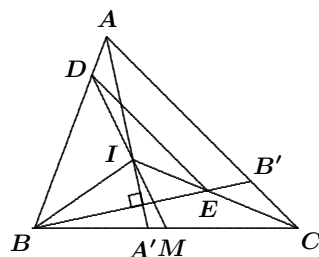
Given triangle ABC with $AB < AC$, let I be its incentre and let M be the mid-point of BC . Suppose that D is the intersection of IM with AB and that E is the intersection of CI with the perpendicular from B to AI .

Prove that $DE \parallel AC$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

Let $A' = AI \cap BC$ and $B' = BE \cap AC$. Let the lengths of BC, CA, AB , be a, b, c , respectively. Since $BB' \perp AIA'$, we have $AB = AB' = c$. Hence, $B'C = b - c$. Since EC bisects $\angle B'CB$, we have

$$\frac{BE}{EB'} = \frac{BC}{CB'} = \frac{a}{b-c}.$$



Since $BA' : A'C = AB : AC$, we have $BA' = \frac{ac}{b+c}$. By Menelaus'

Theorem, we have

$$\begin{aligned} \frac{BD}{DA} &= \left(\frac{BM}{MA'} \right) \left(\frac{A'I}{IA} \right) = \left(\frac{\frac{a}{2}}{\frac{a}{2} - BA'} \right) \left(\frac{BA'}{c} \right) \\ &= \left(\frac{1}{1 - \frac{2c}{b+c}} \right) \left(\frac{a}{b+c} \right) = \frac{a}{b+c-2c} = \frac{a}{b-c}. \end{aligned}$$

Thus, $BE : EB' = BD : DA$, which implies that DE is parallel to AC .

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D.J. SMEENK, Zaltbommel, the Netherlands; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

The solvers showed a wide variety of methods of solving this problem, including barycentric coordinates, Cartesian coordinates, trigonometry, and vectors.

2916. [2003 : 106, 109] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $S = A_1A_2A_3A_4$ be a tetrahedron and let M be the Steiner point; that is, the point M is such that $\sum_{j=1}^4 A_jM$ is minimized. Assuming that M is an interior point of S , and denoting by A'_j the intersection of A_jM with the opposite face, prove that

$$\sum_{j=1}^4 A_jM \geq 3 \sum_{j=1}^4 A'_jM.$$

Solution by Michel Bataille, Rouen, France.

The sum $\sum_{j=1}^4 A_j M$, being extremal at the interior point M , satisfies $\text{grad} \left(\sum_{j=1}^4 A_j M \right) = \vec{0}$; that is,

$$\vec{u}_1 + \vec{u}_2 + \vec{u}_3 + \vec{u}_4 = \vec{0}, \quad (1)$$

where \vec{u}_j denotes the unit vector $\frac{\overrightarrow{MA_j}}{MA_j}$ for $j = 1, 2, 3, 4$. [Ed: If we let $M = (x, y, z)$ and $A_j = (x_j, y_j, z_j)$, then the terms of our sum have the form $F_j = \sqrt{(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2}$ and $\text{grad}(F_j) = \vec{u}_j$.]

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the positive real numbers such that

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 1 \\ \text{and} \quad \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 &= M. \end{aligned}$$

From $\sum_{j=1}^4 \alpha_j \overrightarrow{MA_j} = \vec{0}$ and (1), we deduce

$$\begin{aligned} (\alpha_1 MA_1 - \alpha_4 MA_4) \vec{u}_1 + (\alpha_2 MA_2 - \alpha_4 MA_4) \vec{u}_2 \\ + (\alpha_3 MA_3 - \alpha_4 MA_4) \vec{u}_3 = \vec{0}, \end{aligned}$$

which implies that $\alpha_1 MA_1 = \alpha_2 MA_2 = \alpha_3 MA_3 = \alpha_4 MA_4$. Thus, if c denotes this common positive value, we have

$$\sum_{j=1}^4 MA_j = c \sum_{j=1}^4 \frac{1}{\alpha_j}. \quad (2)$$

Furthermore, $(1 - \alpha_j) \overrightarrow{MA'_j} = -\alpha_j \overrightarrow{MA_j}$ for each $j = 1, 2, 3, 4$. For example, considering $j = 1$, the point $\frac{M - \alpha_1 A_1}{1 - \alpha_1} = \frac{\alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4}{1 - \alpha_1}$ is on the line $A_1 M$ as well as in the plane $A_2 A_3 A_4$, and therefore, this point is A'_1 . Thus, $(1 - \alpha_1) A'_1 = M - \alpha_1 A_1$, and hence, we obtain $(1 - \alpha_1) \overrightarrow{MA'_1} = -\alpha_1 \overrightarrow{MA_1}$.

It follows that

$$\sum_{j=1}^4 MA'_j = c \sum_{j=1}^4 \frac{1}{1 - \alpha_j}. \quad (3)$$

From (2) and (3), the requested inequality is equivalent to

$$\sum_{j=1}^4 \frac{1}{1 - \alpha_j} \leq \frac{1}{3} \sum_{j=1}^4 \frac{1}{\alpha_j}. \quad (4)$$

Now, with the help of the AM–HM Inequality, we obtain

$$\frac{1}{1 - \alpha_1} = \frac{1}{\alpha_2 + \alpha_3 + \alpha_4} \leq \frac{1}{9} \left(\frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} \right).$$

Adding this to the corresponding inequalities for $\frac{1}{1-\alpha_2}$, $\frac{1}{1-\alpha_3}$, $\frac{1}{1-\alpha_4}$, we get

$$\sum_{j=1}^4 \frac{1}{1-\alpha_j} \leq 3 \times \frac{1}{9} \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} \right),$$

and (4) follows.

Also solved by LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Zhou comments that the result may be extended to an n -simplex $S_n = A_1 A_2 \cdots A_{n+1}$ with the conclusion $\sum_{i=1}^{n+1} A_i M \geq n \sum_{i=1}^{n+1} A_i' M$.

2917★. [2003 : 107, 109] Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

If $x_1, x_2, x_3, x_4, x_5 \geq 0$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 1$, prove or disprove that

$$\frac{x_1}{1+x_2} + \frac{x_2}{1+x_3} + \frac{x_3}{1+x_4} + \frac{x_4}{1+x_5} + \frac{x_5}{1+x_1} \geq \frac{5}{6}.$$

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

The statement is false. If we let $x_1 = .5, x_2 = .4, x_3 = .05, x_4 = .03$, and $x_5 = .02$, then $x_1, x_2, x_3, x_4, x_5 \geq 0$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 1$, but

$$\begin{aligned} & \frac{x_1}{1+x_2} + \frac{x_2}{1+x_3} + \frac{x_3}{1+x_4} + \frac{x_4}{1+x_5} + \frac{x_5}{1+x_1} \\ &= \frac{.5}{1.4} + \frac{.4}{1.05} + \frac{.05}{1.03} + \frac{.03}{1.02} + \frac{.02}{1.5} = .829384 \dots < \frac{5}{6}. \end{aligned}$$

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany.

Specht reports numerical experiments that suggest the minimum is obtained when two successive values x_k and x_{k+1} are zero. Under this assumption the minimum value is approximately 0.8161123672.

2918. [2004 : 107, 110] Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Let a_1, a_2, \dots, a_{100} be real numbers satisfying:

$$\begin{aligned} a_1 &\geq a_2 \geq \dots \geq a_{100} \geq 0; \\ a_1^2 + a_2^2 &\geq 200; \\ a_3^2 + a_4^2 + \dots + a_{100}^2 &\geq 200. \end{aligned}$$

What is the minimum value of $a_1 + a_2 + \dots + a_{100}$?

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposers.

First we note that $a_2 > 0$; for otherwise, $a_3 = a_4 = \dots = a_n = 0$ and the condition $a_3^2 + a_4^2 + \dots + a_{100}^2 \geq 200$ fails.

Since $a_2 \geq a_j$ for $j = 3, 4, \dots, 100$, we have

$$a_2(a_3 + a_4 + \dots + a_{100}) \geq a_3^2 + a_4^2 + \dots + a_{100}^2 \geq 200,$$

which implies that $a_3 + a_4 + \dots + a_{100} \geq 200/a_2$. Using this inequality, the fact that $a_1 \geq a_2$, and the AM–GM Inequality, we obtain

$$\begin{aligned} a_1 + a_2 + \dots + a_{100} &\geq a_1 + a_2 + \frac{200}{a_2} \geq 2a_2 + \frac{200}{a_2} \\ &\geq 2\sqrt{(2a_2)\frac{200}{a_2}} = 40. \end{aligned}$$

Inspecting the inequalities above, we find that $a_1 + a_2 + \dots + a_{100}$ attains its minimum value of 40 if and only if $a_1 = a_2 = a_3 = a_4 = 10$ and $a_5 = a_6 = \dots = a_{100} = 0$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria. There was one incomplete solution.

Hess suggested the following generalization of the problem: If a_1, a_2, \dots, a_n are real numbers, find the minimum value of $a_1 + a_2 + \dots + a_n$ subject to the conditions

$$\begin{aligned} a_1 &\geq a_2 \geq \dots \geq a_n \geq 0; \\ a_1^2 + a_2^2 + \dots + a_m^2 &\geq A; \\ a_{m+1}^2 + a_{m+2}^2 + \dots + a_n^2 &\geq B. \end{aligned}$$

He then proceeded to discuss some particular cases of the general question.

2919★. [2004 : 107, 110] *Proposed by Ross Cressman, Wilfrid Laurier University, Waterloo, ON.*

Let $n \in \mathbb{N}$ with $n > 1$, and let

$$T_n = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_j > 0 \text{ for } j = 1, \dots, n, \text{ and } \sum_{j=1}^n x_j = 1 \right\}.$$

Let $p, q, r \in T_n$ such that $\sum_{j=1}^n \sqrt{q_j r_j} < \sum_{j=1}^n \sqrt{p_j r_j}$.

Prove or disprove:

(a) $\sum_{j=1}^n \sqrt{q_j (r_j + p_j)} < \sum_{j=1}^n \sqrt{p_j (r_j + p_j)},$

(b) for all $\lambda \in [0, 1]$,

$$\sum_{j=1}^n \sqrt{q_j (\lambda r_j + (1 - \lambda)p_j)} < \sum_{j=1}^n \sqrt{p_j (\lambda r_j + (1 - \lambda)p_j)}.$$

[Proposer's remarks: (a) is the special case of (b) with $\lambda = \frac{1}{2}$. This question is connected with properties of the Shahshahani metric on T_n , a metric important for population genetics.]

Solution by Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON.

We show that (b) is true when $n = 2$, which implies that (a) is also true when $n = 2$. We then show how to construct examples to prove that (a) and (b) are false when $n \geq 3$.

For $p, q \in \mathbb{R}^n$, let $p \cdot q$ denote the standard inner product in \mathbb{R}^n . Let

$$S_n = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_j > 0 \text{ for } j = 1, \dots, n, \text{ and } \sum_{j=1}^n x_j^2 = 1 \right\}.$$

Part (b) is equivalent to the proposition below:

Proposition: Let $n \in \mathbb{N}$ with $n > 1$. If $p, q, r \in S_n$ such that $q \cdot r < p \cdot r$, and if $s \in S_n$ with components given by $s_j = \sqrt{\lambda r_j^2 + (1 - \lambda)p_j^2}$ for some $\lambda \in [0, 1]$, then $q \cdot s < p \cdot s$.

We first show that the proposition is true for $n = 2$. For convenience, we regard each $p \in \mathbb{R}^n$ both as a vector and as a point. When $n = 2$, we see that p, q, r, s are all on the unit circle and in the interior of the first quadrant. Let \widehat{pq} denote the acute angle between vectors p and q . Then $q \cdot r < p \cdot r$ if and only if $\cos(\widehat{pr}) > \cos(\widehat{qr})$, or equivalently, $\widehat{pr} < \widehat{qr}$, which is true if and only if r is closer to p than to q . Since $\min\{p_j, r_j\} \leq s_j \leq \max\{p_j, r_j\}$ for all j , we see that s is on the arc between p and r . Thus, s is closer to p than to q ; hence, $q \cdot s < p \cdot s$.

Now we construct an example to show that the proposition above is false for $n = 3$. Take $p = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$, $r = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and $\lambda = \frac{1}{2}$. Then p, r , and $s = \left(\frac{1}{2}, \frac{\sqrt{5}}{2\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are all on the unit sphere and in the interior of the first octant; that is, they are all in S_3 . Note that s is not on the great circle that goes through p and r , since these three points are all on the same latitude with $x_3 = \frac{1}{\sqrt{3}}$.

Consider the circle C_1 on S_3 with centre r passing through the point p . (Note that C_1 need not lie entirely on S_3 since S_3 is only $\frac{1}{8}$ of the unit sphere.) This circle divides S_3 into two regions, one "inside" the circle (think of a polar cap) and one "outside". Similarly, the circle C_2 on S_3 with centre s passing through the point p divides S_3 into two regions. Since p, r, s do not lie on a great circle, C_1 and C_2 are not tangent at p . Thus, there are points on S_3 arbitrarily close to p that are outside C_1 and inside C_2 . Clearly, $q \cdot r < p \cdot r$ if and only if q is outside C_1 , and $q \cdot s > p \cdot s$ if and only if q is inside C_2 . Hence, it suffices to take q to be any such point.

In closing, we remark that this method works for any fixed $\lambda \in (0, 1)$ as well, and can be further extended to any $n > 3$.

2920. [2004 : 108, 110] *Proposed by Simon Marshall, student, Onslow College, Wellington, New Zealand.*

Let a , b , and c be positive real numbers. Prove that

$$a^4 + b^4 + c^4 + 2(a^2b^2 + b^2c^2 + c^2a^2) \geq 3(a^3b + b^3c + c^3a).$$

Composite of almost identical solutions by Vasile Cîrtoaje, University of Ploiesti, Romania; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Note that the given inequality is equivalent to

$$(a^2 + b^2 + c^2)^2 - 3(a^3b + b^3c + c^3a) \geq 0.$$

Let $E = (a^2 + b^2 + c^2)^2 - 3(a^3b + b^3c + c^3a)$. Let

$$a = x, \quad b = x + p, \quad \text{and} \quad c = x + q.$$

Then, after some simplifications, we find that

$$E = E(x) = (p^2 - pq + q^2)x^2 - (p^3 - 5p^2q + 4pq^2 + q^3)x + (p^4 - 3p^3q + 2p^2q^2 + q^4).$$

Regarding $E(x)$ as a quadratic in x , its discriminant is

$$\begin{aligned} \Delta &= (p^3 - 5p^2q + 4pq^2 + q^3)^2 \\ &\quad - 4(p^2 - pq + q^2)(p^4 - 3p^3q + 2p^2q^2 + q^4) \\ &= -3(p^6 - 2p^5q - 3p^4q^2 + 6p^3q^3 + 2p^2q^4 - 4pq^5 + q^6) \\ &= -3(p^3 - p^2q - 2pq^2 + q^3)^2. \end{aligned}$$

Since $\Delta \leq 0$ and $p^2 - pq + q^2 \geq 0$, it follows that $E(x) \geq 0$ for all real x .

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer. There were six incorrect solutions and one incomplete solution. Four of the incorrect solutions made the convenient assumption that $a \leq b \leq c$, which is technically incorrect, since the right side of the given inequality possesses only cyclic symmetry rather than complete symmetry. Three solutions gave (without proof) the wrong statement that equality holds if and only if $a = b = c$, even though the given problem did not ask for the determination of the equality cases.

It is clear from the featured solution that the assumption that a , b , and c are positive is superfluous. But, strangely enough, no one pointed this out explicitly.

The proposer gave a more complicated and sophisticated proof using complex numbers and primitive roots of unity. Using this approach, he was able to determine that the equality holds if and only if either $a = b = c$ or, for some constant k , we have $a = k(1 + z + \bar{z})$, $b = k(1 + \omega z + \omega^2 \bar{z})$, and $c = k(1 + \omega^2 z + \omega \bar{z})$ (in any cyclic order), where $\omega = e^{2\pi i/3}$, $z = \frac{1}{\sqrt{7}} \cos \theta$, and $\theta = -\frac{1}{3} \text{Arg}(2 + 6\omega) = -\frac{1}{3} \text{Arg}(-1 + 3\sqrt{3}i)$. If we take $k = 1$ and use the approximate value $\theta \approx 33.63^\circ$, we see that $(a, b, c) \approx (1.629, 1.048, 0.323)$ or any cyclic permutation thereof.

Mihály Bencze, Brasov, Romania, sent in a comment indicating that in one of the 1997 issues of *Octogon Mathematical Magazine*, he proved the following more general inequality: If $x_k > 0$ for $k = 1, 2, \dots, n$ and $\alpha, \beta > 0$, then

$$\alpha \sum_{k=1}^n x_k^4 + \beta \sum_{k=1}^n x_k^2 x_{k+1}^2 \geq (\alpha + \beta) \sum_{k=1}^n x_k^3 x_{k+1},$$

where $x_{k+1} = x_1$. The current problem is the special case when $n = 3$, $\alpha = 1$, and $\beta = 2$.

Cîrtoaje and another solver remarked that the given inequality follows from the identity

$$\begin{aligned} 4(a^2 + b^2 + c^2 - ab - bc - ca)((a + b + c)^2 - 3(a^3b + b^3c + c^3a)) \\ = (A - 5B + 4C)^2 + 3(A - B - 2C + 2D)^2, \end{aligned}$$

where $A = a^3 + b^3 + c^3$, $B = a^2b + b^2c + c^2a$, $C = ab^2 + bc^2 + ca^2$, and $D = 3abc$. Clearly, verifying this identity by hand is a formidable task. The editor turned to MAPLE for help and found the identity to be correct. Unfortunately, the solvers did not explain how they arrived at this amazing identity.

Cîrtoaje mentioned that the given inequality has appeared before as problem #22694 on page 287 of *Gazeta Matematica* 7-8, 1992 (in Romanian), in the following (equivalent) form: If a, b , and c are real numbers, then

$$a^2(a - b)(a - 2b) + b^2(b - c)(b - 2c) + c^2(c - a)(c - 2a) \geq 0.$$

However, he stated that a solution was not published there.

2921. [2004 : 108, 111] Proposed by Barry R. Monson, University of New Brunswick, Fredericton, NB; and J. Chris Fisher, University of Regina, Regina, SK.

These days, with *Cinderella*TM and the *Lénárt sphere*TM at hand, one can do actual spherical constructions, using a spherical ruler to draw the complete great circle through points A and B , and spherical compasses to draw the circle with centre A and radius BC ($\leq \frac{\pi}{2}$, say, on a unit sphere).

Give a simple spherical construction for the vertices of a regular icosahedron inscribed in the sphere.

Solution by the proposers.

The perpendiculars, mid-points, angle bisectors, and circumcircle that we need here can all be constructed by mimicking planar constructions. First draw (the great circle) AX through two non-antipodal points A and X ; then erect the perpendiculars at A and X , meeting at Z , say (one of the two poles for AX). Suppose that AZ , which already has been drawn, meets AX again at B (antipodal to A). At Z erect the perpendicular to AZ , meeting AXB at antipodes Y and W . (The sphere has now been subdivided into congruent octants.)

Find the mid-point K of the quarter great circle AZ , and construct the circle μ with centre K and radius $KZ = KA$. Next construct the circle ν through Z, B, Y , and suppose ν meets μ at V (as well as at Z). Finally, suppose ZV meets AXB at antipodes P (closest to A) and Q .

Claim. A and P are neighbouring vertices of an inscribed icosahedron.

Proof of the claim: Look at where the planes supporting the various circles of the construction meet the ‘equatorial plane’ $AYBW$. The sphere itself traces the equator λ through these points, a circle whose centre M is the mid-point of the Euclidean line segment AB . But the planes determined by μ and ν trace lines (in the equatorial plane) meeting at a point D , so that $\triangle ABD$ is an isosceles right triangle, with DA tangent to λ at A . It is now easy to see that $APBQ$ is a golden rectangle; in fact, that is the content of **CRUX with MAYHEM** problem 2813 [2003 : 47; 2004 : 63–64], namely *If M is the mid-point of side AB of the square $ABCD$, while P (inside the square) and Q are the intersection points of the line MD with the circle centred at M whose radius is MA ($= MB$), then $APBQ$ is a golden rectangle— $PB : PA = (\sqrt{5} + 1)/2$. Therefore, A , P , B , and Q are four of the twelve vertices of a regular icosahedron. (See Coxeter, *Introduction to Geometry*, section 11.2.) The others are similarly located on the great circle bisectors of $\angle AZP$ and $\angle PZB$.*

*No other solutions were submitted, but Zhou offered the comment that the construction is quite simple if one wants to work in a plane: construct the angle $\cos^{-1}(1/\sqrt{5})$ [by means of a $(1, 2, \sqrt{5})$ -right triangle such as $\triangle DAM$ in the above solution]. Since this angle is the central angle subtended by adjacent vertices of the icosahedron, one only has to transfer to the sphere the chord of the unit circle which subtends that angle. This is essentially the way Euclid constructed the Platonic solids. Note that the original problem restricts the construction to the sphere’s surface; it is only for the proof that we are permitted to embed the sphere in Euclidean space. Perhaps readers are happy with life in the Euclidean plane and find it hard to imagine what it would be like living on a spherical surface. In Bieberbach’s *Theorie der geometrischen Konstruktionen*, there is a short section on spherical constructions. It provides general arguments using stereographic projection to show that the spherical possibilities more or less amount to the Euclidean possibilities.*

2922. [2003 : 108, 111] *Proposed by Michel Bataille, Rouen, France.*

Suppose that n is a non-negative integer. Find a closed expression for

$$\sum_{k=0}^n (-1)^k 2^k \binom{n}{k} \binom{2n-k}{n}.$$

Solution by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela.

Let $\langle x^k \rangle f(x)$ denote the coefficient of x^k in the polynomial $f(x)$. Clearly, $\langle x^{n-k} \rangle f(x) = \langle x^n \rangle x^k f(x)$ and $\sum_k \langle x^k \rangle f(x) y^k = f(y)$. Now we have

$$\langle x^k \rangle (1 - 2x)^n = (-2)^k \binom{n}{k} = (-1)^k 2^k \binom{n}{k},$$

and

$$\binom{2n-k}{n} = \binom{2n-k}{n-k} = \langle y^{n-k} \rangle (1+y)^{2n-k} = \langle y^n \rangle y^k (1+y)^{2n-k}.$$

Hence,

$$\begin{aligned}
 & \sum_{k=0}^n (-1)^k 2^k \binom{n}{k} \binom{2n-k}{n} \\
 &= \sum_{k=0}^n \langle x^k \rangle (1-2x)^n \langle y^n \rangle y^k (1+y)^{2n-k} \\
 &= \langle y^n \rangle (1+y)^{2n} \sum_{k=0}^n \langle x^k \rangle (1-2x)^n \left(\frac{y}{1+y} \right)^k \\
 &= \langle y^n \rangle (1+y)^{2n} \left(1 - \frac{2y}{1+y} \right)^n = \langle y^n \rangle (1+y)^n (1-y)^n \\
 &= \langle y^n \rangle (1-y^2)^n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} \binom{n}{n/2} & \text{if } n \text{ is even.} \end{cases}
 \end{aligned}$$

Also solved by CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Janous remarks there is an earlier treatment of this sum, "Unfortunately I have no reference in English, but only one in Russian: Yu. P. Vrazaev, Problems in Algebra and Analysis, Kiev, 1949. [Mimeographed lecture notes]".

2923. [2004 : 108, 111] Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Suppose that $x, y \geq 0$ ($x, y \in \mathbb{R}$) and $x^2 + y^3 \geq x^3 + y^4$. Prove that $x^3 + y^3 \leq 2$.

I. Solution by Chip Curtis, Missouri Southern State College, Joplin, MO, USA.

We proceed by proving the following string of inequalities:

$$x^3 + y^3 \leq x^2 + y^2 \leq x + y \leq 2.$$

Writing

$$x^3 + y^3 = \langle x^{3/2}, y^2 \rangle \cdot \langle x^{3/2}, y \rangle,$$

the Cauchy-Schwarz Inequality, the given inequality, and the AM-GM Inequality imply that

$$\begin{aligned}
 x^3 + y^3 &\leq \sqrt{x^3 + y^4} \cdot \sqrt{x^3 + y^2} \\
 &\leq \sqrt{x^2 + y^3} \cdot \sqrt{x^3 + y^2} \leq \frac{1}{2}(x^2 + y^3 + x^3 + y^2),
 \end{aligned}$$

from which it follows that $x^3 + y^3 \leq x^2 + y^2$.

Similarly,

$$\begin{aligned} x^2 + y^2 &= \langle x^{3/2}, y^{3/2} \rangle \cdot \langle x^{1/2}, y^{1/2} \rangle \leq \sqrt{x^3 + y^3} \cdot \sqrt{x + y} \\ &\leq \sqrt{x^2 + y^2} \cdot \sqrt{x + y} \leq \frac{1}{2}(x^2 + y^2 + x + y), \end{aligned}$$

implying that $x^2 + y^2 \leq x + y$.

Finally,

$$\begin{aligned} x + y &= \langle x, y \rangle \cdot \langle 1, 1 \rangle \leq \sqrt{x^2 + y^2} \cdot \sqrt{2} \\ &\leq \sqrt{x + y} \cdot \sqrt{2} \leq \frac{1}{2}(x + y + 2), \end{aligned}$$

implying that $x + y \leq 2$. Therefore, $x^3 + y^3 \leq 2$.

II. *Generalization by Li Zhou, Polk Community College, Winter Haven, FL, USA, modified slightly by the editor.*

We prove the more general result that if $x, y > 0$ and if r is a real number such that $x^{r-1} + y^r \geq x^r + y^{r+1}$, then $x^r + y^r \leq x^{r-3} + y^{r-3}$. The current problem is the special case when $r = 3$, since the cases when $x = 0$ or $y = 0$ are clearly trivial.

Let $S = x^{r-2}(1-x) + y^{r-1}(1-y)$ and $T = x^{r-1}(1-x) + y^r(1-y)$. Then $T \geq 0$ from the assumption. Since

$$\begin{aligned} S - T &= x^{r-2} - 2x^{r-1} + x^r + y^{r+1} - 2y^r + y^{r-1} \\ &= x^{r-2}(1-x)^2 + y^{r-1}(1-y)^2 \geq 0, \end{aligned}$$

we have $S \geq T \geq 0$. Furthermore, by straightforward algebra, we find that

$$\begin{aligned} x^{r-3} + y^{r-3} - x^r - y^r - x^{r-3}(1-x)^2 - y^{r-3}(1-y^2)^2 \\ &= -x^r - y^r + 2x^{r-2} - x^{r-1} + 2y^{r-1} - y^{r+1} \\ &= 2x^{r-2}(1-x) + 2y^{r-1}(1-y) + x^{r-1} - x^r + y^r - y^{r+1} \\ &= 2S + T. \end{aligned}$$

Hence,

$$\begin{aligned} x^{r-3} + y^{r-3} - x^r - y^r \\ &= 2S + T + x^{r-3}(1-x)^2 + y^{r-3}(1-y^2)^2 \geq 0, \end{aligned}$$

from which $x^r + y^r \leq x^{r-3} + y^{r-3}$ follows immediately.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; and the proposer. There was one incorrect solution.

2924. [2003 : 108, 111] *Proposed by Todor Mitev, University of Rouse, Rouse, Bulgaria.*

Suppose that x_1, \dots, x_n ($n \geq 3$) are positive real numbers satisfying

$$\frac{1}{1 + x_2^2 x_3 \cdots x_n} + \frac{1}{1 + x_1 x_3^2 \cdots x_n} + \cdots + \frac{1}{1 + x_1^2 x_2 \cdots x_{n-1}} \geq \alpha,$$

for some $\alpha > 0$. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} \geq \frac{n\alpha}{n - \alpha} x_1 x_2 \cdots x_n.$$

Solution Li Zhou, Polk Community College, Winter Haven, FL, USA.

The given condition implies that $n > \alpha$. Let

$$t_k = \left(\frac{x_k}{x_{k+1}} \right) \left(\frac{1}{x_1 x_2 \cdots x_n} \right),$$

where indices are taken modulo n . Since $f(t) = \frac{t}{t+1}$ is a concave function on $(0, \infty)$, we have

$$\begin{aligned} \frac{n(t_1 + t_2 + \cdots + t_n)}{t_1 + t_2 + \cdots + t_n + n} &= n f\left(\frac{t_1 + t_2 + \cdots + t_n}{n}\right) \\ &\geq f(t_1) + f(t_2) + \cdots + f(t_n) \\ &= \frac{1}{1 + t_1^{-1}} + \frac{1}{1 + t_2^{-1}} + \cdots + \frac{1}{1 + t_n^{-1}} \geq \alpha, \end{aligned}$$

which is equivalent to

$$t_1 + t_2 + \cdots + t_n \geq \frac{n\alpha}{n - \alpha}.$$

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

2925. [2004 : 109, 111] *Proposed by Michel Bataille, Rouen, France.*

Let n be an integer with $n \geq 3$. Determine the zeroes of the function

$$f_n(x) = \sum_{k=1}^{n-1} \frac{\sin(k\pi/n)}{\sin((k\pi/n) - x)}.$$

Solution by the proposer.

It is quite easy to check that the zeroes of $\sin x$ are not zeroes of $f_n(x)$. Therefore, we can impose the restriction $\sin x \neq 0$. Then we may write

$$(\sin x)f_n(x) = \sum_{k=1}^{n-1} \frac{1}{\cot x - \cot(k\pi/n)} = \frac{P'_{n-1}(\cot x)}{P_{n-1}(\cot x)},$$

where

$$P_{n-1}(z) = \prod_{k=1}^{n-1} (z - \cot(k\pi/n)).$$

Thus, $P_{n-1}(z)$ is a polynomial of degree $n-1$, and its zeroes are the numbers $\cot(k\pi/n)$ for $k \in \{1, 2, \dots, n-1\}$. Observe that these are also the roots of the equation $\left(\frac{z+i}{z-i}\right)^n = 1$. It follows that

$$P_{n-1}(z) = \frac{1}{2in} \left((z+i)^n - (z-i)^n \right).$$

Then $P'_{n-1}(z) = (n-1)P_{n-2}(z)$. The zeroes of $P'_{n-1}(z)$ are the zeroes of $P_{n-2}(z)$, namely, the numbers $\cot\left(\frac{k\pi}{n-1}\right)$ for $k \in \{1, 2, \dots, n-2\}$. These numbers are not zeroes of $P_{n-1}(z)$. Therefore, they are the zeroes of $P'_{n-1}(z)/P_{n-1}(z)$.

We can now conclude that the zeroes of $f_n(x)$ are the values of x such that $\cot x = \cot\left(\frac{k\pi}{n-1}\right)$ for $k \in \{1, 2, \dots, n-2\}$. These are the numbers $\frac{m\pi}{n-1}$, where m is any integer not divisible by $n-1$.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There was one incorrect solution. Janous and Zhou used the identity

$$f_n(x) = \frac{n \sin((n-1)x)}{\sin(nx)},$$

which is valid as long as x is not a zero of $\sin(nx)$. Since the zeroes of the right side are quite evident, this identity provides an immediate solution to the problem.

Zhou supplied a proof of the above identity. An elegant proof may be obtained by going a little further with the ideas in the solution above. We leave this as an exercise for the reader.

2926. [2004 : 172, 174] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In circle Γ with centre O and radius R , we have three parallel chords A_1A_2 , B_1B_2 , and C_1C_2 . Show that the orthocentres of the eight triangles having vertices A_i , B_j , and C_k ($i, j, k \in \{1, 2\}$) are collinear.

Composite of similar solutions by Christopher J. Bradley, Bristol, UK; D. Kipp Johnson, Beaverton, OR, USA; Doug Newman, Lancaster, CA, USA; Joel Schlosberg, Bayside, NY, USA; Toshio Seimiya, Kawasaki, Japan; Peter Y. Woo, Biola University, La Mirada, CA, USA; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; Li Zhou, Polk Community College, Winter Haven, FL, USA; Titu Zvonaru, Comănești, Romania; and the proposer.

Choose coordinates so that the end-points of the given chords are

$$\begin{aligned} A_1(-\cos \alpha, \sin \alpha), & \quad A_2(\cos \alpha, \sin \alpha), \\ B_1(-\cos \beta, \sin \beta), & \quad B_2(\cos \beta, \sin \beta), \\ C_1(-\cos \gamma, \sin \gamma), & \quad C_2(\cos \gamma, \sin \gamma). \end{aligned}$$

Then the eight orthocentres are

$$(\pm \cos \alpha \pm \cos \beta \pm \cos \gamma, \sin \alpha + \sin \beta + \sin \gamma).$$

(This is a known result; see, for example, L.-S. Hahn, *Complex Numbers and Geometry*, Math. Association of America, 1994, p. 71. It is, however, more easily proved than looked up: If P , Q , and R are three vectors from the origin O to points on a circle centred at O , then the centroid M of the triangle formed by the tips of these vectors is at $(P + Q + R)/3$, and the points O , H , M on the Euler line satisfy $OH = 3OM$.)

Our proof is completed by observing that all eight of the orthocentres lie on the line $y = \sin \alpha + \sin \beta + \sin \gamma$, which is parallel to the given chords.

Also solved by MICHEL BATAILLE, Rouen, France; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOSÉ H. NIETO, Universidad del Zulia, Maracaibo, Venezuela; and ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany.

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