THE OLYMPIAD CORNER

No. 211

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Another year has sped past. My thanks go to our faithful readers and contributors for sending us contest materials and solutions:

Mohammed Aassila	Murray S. Klamkin
Arthur Baragar	Marcin E. Kuczma
Michel Bataille	Vedula N. Murty
Francisco Bellot Rosado	Richard Nowakowski
Andrew Blinn	Challa K.S.N. Murali Sankar
Miguel Amengual Covas	Heinz-Jürgen Seiffert
Pierre Bornsztein	Toshio Seimiya
Jacqueline Freeman	Achilleas Sinefakopoulos
J.P. Grossman	Panos E. Tsaoussoglou
Walther Janous	Edward T.H. Wang
Geoffrey A. Kandall	Sam Wong

Special thanks also go to Joanne Longworth who transforms my scribbles into a TEX file that is usually under tight time line restrictions, but always with good humour and great skill.

We begin the New Year with the problems of the Ukrainian Mathematical Olympiad, March 1997. Thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina for collecting the problems for our use in the *Corner*.

UKRAINIAN MATHEMATICAL OLYMPIAD
Selected Problems
March 1997

1. (9th Grade) Cells of some rectangular board are coloured as chess board cells. In each cell an integer is written. It is known that the sum of numbers in each row is even and the sum of numbers in each column is even. Prove that the sum of all numbers in the black cells is even.

2. (10th Grade) Solve the system in real numbers

$$\left\{\begin{array}{rcl} x_1+x_2+\cdots+x_{1997} & = & 1997 \\ x_1^4+x_2^4+\cdots+x_{1997}^4 & = & x_1^3+x_2^3+\cdots+x_{1997}^3 \end{array}\right.$$

3. (10th Grade) Let $d(n)$ denote the greatest odd divisor of the natural number n. We define the function $f : N \to N$ as follows: $f(2n-1) = 2^n$, $f(2n) = n + \frac{2n}{d(n)}$ for all $n \in N$.
Find all k such that $f(f(\ldots f(1)\ldots)) = 1997$ where f is iterated k times.

4. (10th Grade) In space two regular pentagons $ABCDE$ and $AEKPL$ are situated so that $\angle DAK = 60°$.
Prove that the planes (ACK) and (BAL) are perpendicular.

5. (11th Grade) It is known that the equation $ax^3 + bx^2 + cx + d = 0$ with respect to x has three distinct real roots. How many roots does the equation $4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2$ have?

6. (11th Grade) Let $\mathbb{Q}^+$ denote the set of all positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$:

(a) $f(x+1) \; = \; f(x) + 1$,

(b) $f(x^2) \; = \; (f(x))^2$.

7. (11th Grade) Find the smallest n such that among any n integers there are 18 integers whose sum is divisible by 18.

8. (11th Grade) At the edges AB, BC, CD, DA of parallelepiped $ABCDA_1B_1C_1D_1$ (not necessarily right) points K, L, M, N respectively are taken. Prove that the four centres of the escribed spheres of A_1AKN, B_1BKL, C_1CLM, D_1DMN are vertices of a parallelogram.

Next we give the problems of the two papers of the Tenth Irish Mathematical Olympiad 1997. Thanks again go to Richard Nowakowski for collecting them for us when he was Canadian Team Leader at the IMO in Argentina.

TENTH IRISH MATHEMATICAL OLYMPIAD
May 10, 1997
First Paper

1. Find (with proof) all pairs of integers (x, y) satisfying the equation

$$1 + 1996x + 1998y \; = \; xy\,.$$

2. Let ABC be an equilateral triangle. For a point M inside ABC, let D, E, F be the feet of the perpendiculars from M onto BC, CA, AB, respectively. Find the locus of all such points M for which $\angle FDE$ is a right angle.

3. Find all polynomials $p(x)$ satisfying the equation

$$(x-16)p(2x) \; = \; 16(x-1)p(x)$$

for all x.

4. Let a, b, c be non-negative real numbers. Suppose that $a+b+c \geq abc$. Prove that $a^2+b^2+c^2 \geq abc$.

5. Let S be the set of all odd integers greater than one. For each $x \in S$, denote by $\delta(x)$ the unique integer satisfying the inequality

$$2^{\delta(x)} \; < \; x \; < \; 2^{\delta(x)+1} \, .$$

For a, $b \in S$, define

$$a * b \; = \; 2^{\delta(a)-1}(b-3)+a \, .$$

[For example, to calculate $5*7$ note that $2^2 < 5 < 2^3$, so that $\delta(5)=2$, and hence, $5*7 = 2^{2-1}(7-3)+5 = 13$. Also $2^2 < 7 < 2^3$, so that $\delta(7)=2$ and $7*5 = 2^{2-1}(5-3)+7 = 11$].

Prove that if a, b, $c \in S$, then

(a) $a*b \in S$ and

(b) $(a*b)*c \; = \; a*(b*c)$.

6. Given a positive integer n, denote by $\sigma(n)$ the sum of all the positive integers which divide n. [For example, $\sigma(3) = 1+3 = 4$, $\sigma(6) = 1+2+3+6 = 12$, $\sigma(12) = 1+2+3+4+6+12 = 28$].

We say that n is abundant if $\sigma(n) > 2n$. (Thus, for example, 12 is abundant). Let a, b be positive integers and suppose that a is abundant. Prove that ab is abundant.

7. $ABCD$ is a quadrilateral which is circumscribed about a circle Γ (that is, each side of the quadrilateral is tangent to Γ). If $\angle A = \angle B = 120°$, $\angle D = 90°$ and BC has length 1, find, with proof, the length of AD.

8. Let A be a subset of $\{0, 1, 2, 3, \ldots, 1997\}$ containing more than 1000 elements. Prove that either A contains a power of 2 (that is, a number of the form 2^k with k a non-negative integer) or there exist two distinct elements $a, b \in A$ such that $a+b$ is a power of 2.

9. Let S be the set of all natural numbers n satisfying the following conditions:

(a) n has 1000 digits

(b) all the digits of n are odd and

(c) the absolute value of the difference between adjacent digits of n is 2.

Determine the number of distinct elements of S.

10. Let p be a prime number and n a natural number and let $T = \{1, 2, 3, \ldots, n\}$. Then n is called *p-partitionable* if there exist non-empty subsets $T_1, T_2, \ldots, T_p$ of T such that

(i) $T = T_1 \cup T_2 \cup \cdots \cup T_p$

(ii) $T_1, T_2, \ldots, T_p$ are disjoint (that is $T_i \cap T_j$ is the empty set for all i, j with $i \neq j$) and

(iii) the sum of the elements in T_i is the same for $i = 1, 2, \ldots, p$.

[For example, 5 is 3-partitionable since, if we take $T_1 = \{1, 4\}$, $T_2 = \{2, 3\}$, $T_3 = \{5\}$, then (i), (ii) and (iii) are satisfied. Also 6 is 3-partitionable since, if we take $T_1 = \{1, 6\}$, $T_2 = \{2, 5\}$, $T_3 = \{3, 4\}$, then (i), (ii) and (iii) are satisfied].

(a) Suppose that n is p-partitionable. Prove that p divides n or $n + 1$.

(b) Suppose that n is divisible by $2p$. Prove that n is p-partitionable.

As a third contest this number, we give the problems of the Hungary-Israel Bi-national Mathematics Competition, 1997. Thanks go to Richard Nowakowski for collecting the problems for use in the *Corner* when he was Team Leader to the IMO in Argentina.

THE HUNGARY-ISRAEL BI-NATIONAL MATHEMATICAL COMPETITION 1997

April 13–20, 1997

Time: 4.5 hours

1. Is there an integer N such that

$$(\sqrt{1997} - \sqrt{1996})^{1998} = \sqrt{N} - \sqrt{N-1}\,?$$

2. Find all real numbers α with the following property: for any positive integer n there exists an integer m such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{3n}\,.$$

3. ABC is an acute angled triangle whose circumcentre is O. The intersection points of the diameters of the circumcircle, passing through A, B, C, with the opposite sides are A_1, B_1, C_1 respectively. The circumradius of the triangle ABC is of length $2p$ where p is a prime. The lengths OA_1, OB_1, OC_1 are integers. What are the lengths of the sides of the triangle?

4. What is the number of distinct sequences of length 1997 that can be formed by using the letters A, B, C, where each letter appears an odd number of times?

5. The three squares ACC_1A'', ABB_1A', $BCDE$ are constructed on the sides of a given triangle ABC, outwards. The centre of the square $BCDE$ is P. Prove that the three lines $A'C$, $A''B$ and PA pass through one point.

6. Can a closed disk be decomposed into a union of two congruent parts having no common points?

As a fourth contest to resharpen your problem solving skills we give the problems of the 10$^{\text{th}}$ Grade, 36$^{\text{th}}$ Armenian National Olympiad in Mathematics. Thanks go to Richard Nowakowski for collecting the problems for use in the *Corner* when he was Team Leader to the IMO in Argentina.

36$^{\text{th}}$ ARMENIAN NATIONAL OLYMPIAD IN MATHEMATICS

Republic of Armenia, 1997

10$^{\text{th}}$ Grade

1. Let

$$p(x) \;=\; (x-a_1)^{n_1}(x-a_1)^{n_2}(x-a_3)^{n_3}$$

be a polynomial, such that

$$p(x)-1 \;=\; (x-b_1)^{k_1}(x-b_2)^{k_2}(x-b_3)^{k_3}\,,$$

where the numbers a_1, a_2, a_3, as well as b_1, b_2, b_3, are distinct and n_1, n_2, n_3, k_1, k_2, k_3 are natural numbers. Prove, that the degree of the polynomial $p(x)$ does not exceed 5.

2. Suppose a and b are natural numbers, such that $(a+b)$ is an odd number. Prove that for any division of the set of natural numbers into two groups, there will be two numbers from the same group, the difference of which is either a or b.

3. Prove that, for any points A, B, C, D, E, F, the following inequality holds:

$$AD^2+BE^2+CF^2 \;\le\; 2(AB^2+BC^2+CD^2+DE^2+EF^2+FA^2)\,.$$

4. It is known that the function $f(x)$ is defined on the set of natural numbers, taking values from the natural numbers, and that it satisfies the following conditions:

(a) $f(xy)=f(x)+f(y)-1$ for any x, $y\in N$,

(b) the equality $f(x)=1$ is true for finitely many numbers,

(c) $f(30)=4$.

Find $f(14400)$.

5. The rectangle P_1 is inscribed in the rectangle P_2 which has sides c, d ($c \leq d$), where d is less than the long side of P_1. Prove that, for any angle α between two lines, containing any two sides of rectangles P_1 and P_2, the following inequality holds:

$$\sin 2\alpha \;\geq\; \frac{c}{d}\,.$$

Now we turn to solutions by our readers to problems posed in the *Corner*. First a nice alternate solution to a problem treated in 1999.

3. [1998 : 69; 1999 : 201]

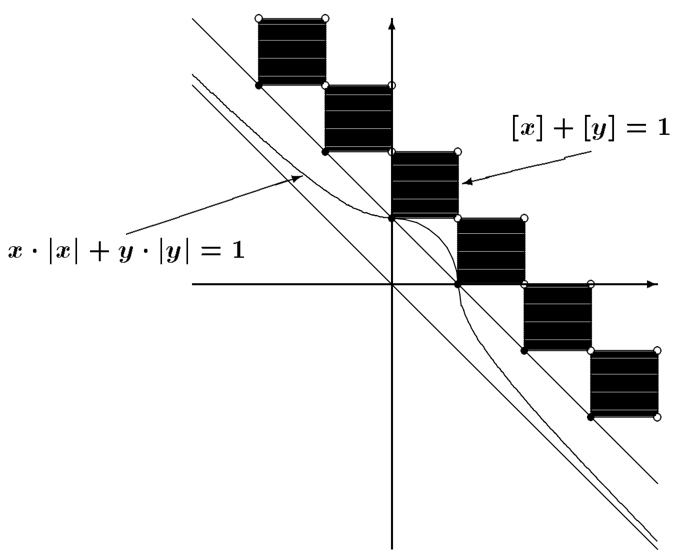
Solve the following system of equations:

$$x \cdot |x| + y \cdot |y| \;=\; 1, \quad [x] + [y] \;=\; 1\,,$$

in which $|t|$ and $[t]$ represent the absolute value and the integer part of the real number t.

Alternate Solution by Michel Bataille, Rouen, France.

It has been shown that the only solutions are $(x, y) = (1, 0)$ and $(0, 1)$ [1999 : 201]. Here is an alternative proof 'without words' ...



Now we turn to reader solutions to problems of the April 1999 *Corner*. The first group of solutions are to problems of the 47^{th} Polish Mathematical Olympiad [1999 : 132–133].

1. Find all pairs (n, r), with n a positive integer, r a real number, for which the polynomial $(x+1)^n - r$ is divisible by $2x^2+2x+1$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Bataille (although they were all very similar).

By division, we can find a polynomial $q(x)$ and two real numbers a, b such that:

$$(x+1)^n - r \;=\; (2x^2+2x+1)\cdot q(x) + ax + b\,.$$

Since the complex roots of $2x^2+2x+1$ are $-\frac{1}{2}+\frac{i}{2}$ and $-\frac{1}{2}-\frac{i}{2}$, the numbers a, b satisfy the system:

$$\begin{cases} a\left(-\frac{1}{2}+\frac{i}{2}\right)+b &=& \left(1+\frac{-1+i}{2}\right)^n - r \\ a\left(-\frac{1}{2}-\frac{i}{2}\right)+b &=& \left(1+\frac{-1-i}{2}\right)^n - r\,. \end{cases}$$

Taking into account:

$$1+\frac{-1+i}{2} \;=\; \frac{1+i}{2} \;=\; \frac{1}{\sqrt{2}}e^{i\pi/4}$$

and

$$1+\frac{-1-i}{2} \;=\; \frac{1-i}{2} \;=\; \frac{1}{\sqrt{2}}e^{-i\pi/4}\,,$$

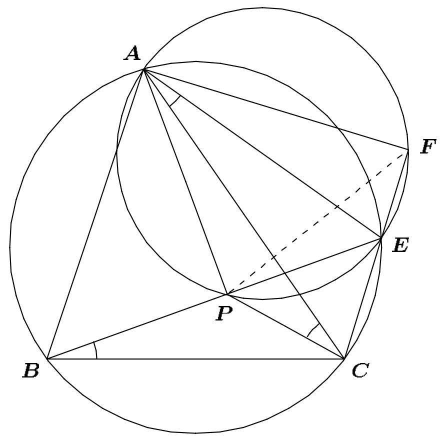
we get, by subtraction:

$$ai \;=\; 2^{-n/2}(e^{in\pi/4}-e^{-in\pi/4}) \;=\; 2^{-n/2}\cdot 2i\sin(n\pi/4)\,.$$

Hence, $a=0$ if and only if $\sin(n\pi/4)=0$; that is, n is a multiple of 4. Assuming that $n=4k$, where k is a positive integer, we obtain that $b = 2^{-2k}(e^{i\pi})^k - r$, so that $b=0$ if and only if $r=\frac{(-1)^k}{4^k}$. Thus, the solutions are the pairs $\left(4k, \frac{(-1)^k}{4^k}\right)$ where k is a positive integer.

2. Given is a triangle ABC and a point P inside it satisfying the conditions: $\angle PBC = \angle PCA < \angle PAB$. Line BP cuts the circumcircle of ABC at B and E. The circumcircle of triangle APE meets line CE at E and F. Show that the points A, P, E, F are consecutive vertices of a quadrilateral. Also show that the ratio of the area of quadrilateral $APEF$ to the area of triangle ABP does not depend on the choice of P.

Solution by Toshio Seimiya, Kawasaki, Japan.



Since

$$\begin{aligned}\angle PFC &= \angle PAE = \angle PAC + \angle EAC = \angle PAC + \angle EBC \\ &= \angle PAC + \angle PBC \\ &< \angle PAC + \angle PAB = \angle BAC = \angle BEC = \angle PEC\,.\end{aligned}$$

Thus, F is a point on CE beyond E.

Therefore, A, P, E, F are consecutive vertices of a quadrilateral. Since $\angle EAC = \angle EBC = \angle PBC = \angle PCA$, we have $PC \| AE$. Thus, we have $[APE] = [ACE]$, where $[A_1A_2 \ldots A_n]$ denotes the area of n-gon A_1, A_2, $\ldots$, A_n.

Hence,

$$[APEF] = [APE] + [AEF] = [ACE] + [AEF] = [ACF]\,.$$

Since

$$\angle ACF = \angle ACE = \angle ABE = \angle ABP\,,$$

and

$$\angle AFC = \angle AFE = \angle APB\,,$$

we have $\triangle ACF \sim \triangle ABP$.

Thus, we get $[ACF] : [ABP] = AC^2 : AB^2$; that is

$$[APEF] : [ABP] = AC^2 : AB^2 \quad (= \text{constant}).$$

Therefore, $[APEF] : [ABP]$ does not depend on the choice of P.

3. Let $n \geq 2$ be a fixed natural number and let $a_1, a_2, \ldots, a_n$ be positive numbers whose sum equals 1.

(a) Prove the inequality

$$2\sum_{i<j} x_i x_j \;\leq\; \frac{n-2}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i}$$

for any positive numbers $x_1, x_2, \ldots, x_n$ summing to 1.

(b) Determine all n-tuples of positive numbers $x_1, x_2, \ldots, x_n$ summing to 1 for which equality holds.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; by George Evagelopoulos, Athens, Greece; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Klamkin's solution.

(a) Replacing a_i in the numerator by $(a_i - 1) + 1$ and simplifying, we get

$$1 \;\leq\; (n-1)\sum \frac{x_i^2}{1-a_i} \;=\; \left(\sum(1-a_i)\right)\left(\sum \frac{x_i^2}{1-a_i}\right),$$

where the sums here and subsequently are over $i = 1$ to n. By Cauchy's inequality, the right hand side is $\geq (\sum x_i)^2 = 1$.

(b) There is equality if and only if $x_i = k(1-a_i)$ where $k = \frac{1}{n-1}$.

4. Let $ABCD$ be a tetrahedron with

$$\angle BAC \;=\; \angle ACD \quad \text{and} \quad \angle ABD \;=\; \angle BDC\,.$$

Show that edges AB and CD have equal lengths.

Solutions by Michel Bataille, Rouen, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Toshio Seimiya, Kawasaki, Japan. We give Klamkin's solution and remarks.

It is to be noted that there is a tacit assumption the figure is 3-dimensional. The result is not valid if $ABCD$ is a trapezoid with AB parallel to CD.

Letting $b = AB$, $c = AC$, $d = AD$, $b_1 = CD$, $c_1 = BD$, and $d_1 = BC$, and applying the Law of Cosines, we get

$$2c\,\cos\angle BAC \;=\; \frac{b^2+c^2-d_1^2}{b} \;=\; \frac{b_1^2+c^2-d^2}{b_1}\,, \tag{1}$$

$$2c_1\,\cos\angle ABD \;=\; \frac{b_1^2+c_1^2-d_1^2}{b_1} \;=\; \frac{b^2+c_1^2-d^2}{b}\,. \tag{2}$$

Equations (1) and (2) reduce to

$$(bb_1 - c^2)(b - b_1) \;=\; b_1 d_1^2 - bd^2\,, \tag{1'}$$

$$(bb_1 - c_1^2)(b - b_1) \;=\; b_1 d^2 - b d_1^2\,. \qquad (2')$$

Adding the latter two equations, we get

$$(b - b_1)(2bb_1 + d^2 + d_1^2 - c^2 - c_1^2) \;=\; 0\,. \qquad (3)$$

We now show that the second factor of (3) is > 0, so that $b = b_1$. Letting $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ denote the vectors AB, AC, and AD, respectively, the second factor is also given by

$$2|\mathbf{B}||\mathbf{C}-\mathbf{D}| + \mathbf{D}^2 + (\mathbf{B}-\mathbf{C})^2 - \mathbf{C}^2 - (\mathbf{B}-\mathbf{D})^2 \;=\; 2(|\mathbf{B}||\mathbf{C}-\mathbf{D}| - \mathbf{B}(\mathbf{C}-\mathbf{D}))\,.$$

Hence, this factor is > 0 provided AB is not parallel to CD. It now also follows from (1) or (2) that $d = d_1$.

Comment. A related problem in which the angle hypotheses were

$$\angle BAD \;=\; \angle BCD \quad \text{and} \quad \angle ABC \;=\; \angle ADC$$

and the conclusions the same, occurred as problem B-5 in the 1972 William Lowell Putnam Competition. This result is due to Thebault [1] and there is a simpler proof by Bottema [2].

References:
[1] V. Thebault, On the skew quadrilateral, *Amer. Math. Monthly* **60** (1953) 102–105.
[2] O. Bottema, Notes on the skew quadrilateral, *Amer. Math. Monthly* **61** (1954) 692–693.

5. For a natural number $k \geq 1$ let $p(k)$ denote the least prime number which is not a divisor of k. If $p(k) > 2$, define $q(k)$ to be the product of all primes less than $p(k)$, and if $p(k) = 2$, set $q(k) = 1$. Consider the sequence

$$x_0 \;=\; 1\,, \quad x_{n+1} \;=\; \frac{x_n p(x_n)}{q(x_n)} \quad \text{for} \quad n \;=\; 0, 1, 2, \ldots\,.$$

Determine all natural numbers n such that $x_n = 111\,111$.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's solution.

We will prove that $x_n = 111\,111$ if and only if $n = 2106$. We adopt the convention $\prod_{x\in\phi} x = 1$ and $p_1 = 2$, p_i denotes the i^{th} prime (in increasing order).

Then, for every $k \in \mathbb{N}^*$, $q(x) = \prod_{p_i < p(k)} p_i$. We have $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 2 \times 3$, $x_4 = 5$, $x_5 = 2 \times 5$, $x_6 = 3 \times 5$, $x_7 = 2 \times 3 \times 5$, $x_8 = 7$.

By induction on $n \geq 1$, we prove that, if

$$n \;=\; \sum_{i\geq 0} \alpha_i 2^i, \quad \alpha_i \in \{0,1\} \quad \text{(binary expansion of } n\text{)}\,,$$

then

$$x_n = \prod_{i\geq 0} p_{i+1}^{\alpha_i} . \qquad (1)$$

For $n = 1$, $\alpha_0 = 1$ and $\alpha_i = 0$ for $i \geq 1$, then

$$\prod_{i\geq 0} p_{i+1}^{\alpha_i} = 2 = x_1 ,$$

and so, (1) holds for $n = 1$.

Let $n \geq 1$ be a fixed integer. We suppose (1) holds for n,

$$n = \sum_{i\geq 0} \alpha_i 2^i .$$

Let i_0 be the smallest integer such that $\alpha_{i_0} = 0$. Then $n+1 = \sum_{i\geq 0} \beta_i 2^i$ where:

$$\begin{cases} \beta_i = 0 & \text{if } i < i_0 , \\ \beta_{i_0} = 1 , & \\ \beta_i = \alpha_i & \text{if } i > i_0 . \end{cases} \qquad (2)$$

From the induction hypothesis (1) we have:

$$x_n = \prod_{i\geq 0} p_{i+1}^{\alpha_i} = \prod_{i<i_0} p_{i+1} \times \prod_{i>i_0} p_{i+1}^{\alpha_i} \quad (\text{since } \alpha_{i_0} = 0 \text{ and } \alpha_i = 1 \text{ for } i < i_0).$$

Then $p(x_n) = p_{i_0+1}$ and $q(x_n) = \prod_{i<i_0} p_{i+1}$.

It follows that

$$\begin{aligned} x_{n+1} &= \frac{\prod_{i<i_0} p_{i+1} \times \prod_{i>i_0} p_{i+1}^{\alpha_i} \times p_{i_0+1}}{\prod_{i<i_0} p_{i+1}} = p_{i_0+1} \prod_{i>i_0} p_{i+1}^{\alpha_i} \\ &= \prod_{i<i_0} p_{i+1}^{\beta_i} \times p_{i_0+1}^{\beta_{i_0}} \times \prod_{i>i_0} p_{i+1}^{\beta_i} \quad (\text{from } (2)) \\ &= \prod_{i\geq 0} p_{i+1}^{\beta_i} . \end{aligned}$$

Then (1) holds for $n+1$, and the induction is achieved.

Since $x_0 = 1$, we have $x_0 \neq 111\,111$.

For $n \geq 1$, from (1), we deduce:

$$\begin{aligned} x_n = 111\,111 \iff x_n &= 3 \times 7 \times 11 \times 13 \times 37 \\ &= p_2 p_4 p_5 p_6 p_{12} , \\ \iff n &= 2^{11} + 2^5 + 2^4 + 2^3 + 2^1 , \\ \iff n &= 2106 , \quad \text{as claimed.} \end{aligned}$$

Next we turn to solutions from our readers to problems of the 10^{th} Nordic Mathematical Contest [1999 : 133–134].

1. Prove the existence of a positive integer divisible by 1996 the sum of whose decimal digits is 1996.

Solutions by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Bataille.

Let $S(n)$ denote the sum of the decimal digits of the positive integer n. We observe:

$$S(1996) = 25 \quad \text{and} \quad S(2 \times 1996) = S(3992) = 23\,.$$

Since $1996 = 78 \times 25 + 23 + 23$, the integer

$$n = 19961996\ldots 199639923992$$

(where the group of digits 1996 is repeated 78 times) satisfies $S(n) = 1996$. Moreover, $n = 3992 + 3992 \times 10^4 + 1996 \times 10^8 + \cdots + 1996 \times 10^{4\times 79}$ is clearly a multiple of 1996.

2. Determine all real x such that

$$x^n + x^{-n}$$

is an integer for any integer n.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's write-up.

Let x be a non-zero real number and, for any integer n, let $a_n = x^n + x^{-n}$. Since $a_{-n} = a_n$, we can consider only non-negative integers n.

We have

$$a_0 = 2\,;\, a_1 = x + \frac{1}{x}\,; \quad \ldots\,;$$

$$a_{n+1} = x^{n+1} + \frac{1}{x^{n+1}} = \left(x^n + \frac{1}{x^n}\right)\left(x + \frac{1}{x}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right)\,;$$

so that, for $n \geq 1$, we have $a_{n+1} = a_n a_1 - a_{n-1}$.

With the help of this relation, an easy induction shows that a_n is an integer for all positive integers n if (and only if) a_1 is an integer.

Now, the equation $x + \frac{1}{x} = k$, where k is an integer, is equivalent to $x^2 - kx + 1 = 0$. Hence, the suitable x are the real numbers

$$\frac{k + \sqrt{k^2 - 4}}{2} \quad \text{and} \quad \frac{k - \sqrt{k^2 - 4}}{2}\,,$$

where k is any integer such that $|k| \geq 2$.

3. A circle has the altitude from A in a triangle ABC as a diameter, and intersects AB and AC in the points D and E, respectively, different from A. Prove that the circumcentre of triangle ABC lies on the altitude from A in triangle ADE, or it produced.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.

Let H be the foot of the altitude from A to BC. Then

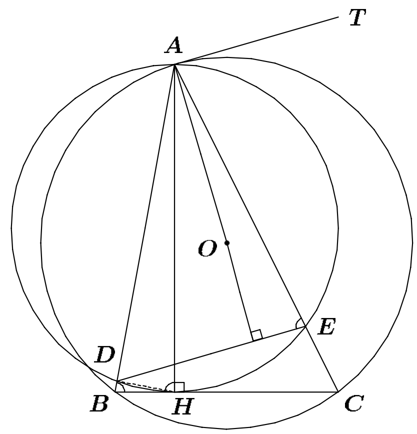
$$AH \perp BC .$$

Since AH is the diameter of the circle $ADHE$, we have

$$\angle ADH = 90^\circ .$$

Since $\angle AHB = 90^\circ$ and $HD \perp AB$, we get

$$\angle AHD = \angle ABH = \angle ABC .$$



Since $\angle AHD = \angle AED$ (because A, D, H, E are concyclic), we have

$$\angle ABC = \angle AED . \tag{1}$$

Let O be the circumcentre of $\triangle ABC$, and let AT be the tangent at A to the circumcircle of $\triangle ABC$. Then

$$\angle TAC = \angle ABC . \tag{2}$$

From (1) and (2) we get $\angle AED = \angle TAC$. Thus, $DE \perp AT$. Since $AT \perp AO$, we have $AO \perp DE$.

Therefore, the perpendicular from A to DE passes through O.

4. A real-valued function f is defined for positive integers, and a positive integer a satisfies

$$f(a) = f(1995), \quad f(a+1) = f(1996), \quad f(a+2) = f(1997),$$

$$f(n+a) = \frac{f(n)-1}{f(n)+1} \quad \text{for any positive integer } n .$$

(a) Prove that $f(n+4a) = f(n)$ for any positive integer n.

(b) Determine the smallest possible value of a.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Aassila's write-up.

(a) From $f(n+a) = \frac{f(n)-1}{f(n)+1}$, we deduce that

$$f(n+2a) \;=\; f((n+a)+a) = \frac{\frac{f(n)-1}{f(n)+1} - 1}{\frac{f(n)-1}{f(n)+1} + 1} \;=\; -\frac{1}{f(n)} \qquad \text{and}$$

$$f(n+4a) \;=\; f((n+2a)+2a) \;=\; -\frac{1}{f(n+2a)} \;=\; f(n)\,.$$

(b) The smallest possible value of a is 3. Indeed, if $a = 1$, then

$$f(1) \;=\; f(a) \;=\; f(1995) \;=\; f(3+498\cdot 4a) \;=\; f(3) \;=\; f(1+2a) \;=\; -\frac{1}{f(1)}\,,$$

and hence,

$$(f(1))^2 \;=\; -1 \;, \quad \text{which is impossible.}$$

If $a = 2$, then

$$\begin{aligned} f(2) \;&=\; f(a) \;=\; f(1995) \;=\; f(3+249\cdot 4a) \;=\; f(3) \;=\; f(a+1) \\ &=\; f(1996) \;=\; f(4+249\cdot 4a) \;=\; f(4) \;=\; f(2+a) \;=\; \frac{f(2)-1}{f(2)+1}\,, \end{aligned}$$

and hence, $(f(2))^2 = -1$, which is impossible.

If $a = 3$, we should be sure that $f(n) \neq -1$ for all $n \in \mathbb{N}$. Let $f(1)$, $f(2)$ and $f(3)$ be chosen arbitrarily different from -1, 0, 1. Then, for $n = 1$, 2 or 3, and since $f(n) \neq -1$, 0, 1, none of $f(n)$, $f(n+3) = \frac{f(n)-1}{f(n)+1}$, $f(n+6) = -\frac{1}{f(n)}$ and $f(n+9) = -\frac{f(n)+1}{f(n)-1}$ is equal to -1, and then, by (a), $f(n) \neq -1$ for all $n \in \mathbb{N}$.

By construction, we have

$$f(n+a) \;=\; f(n+3) \;=\; \frac{f(n)-1}{f(n)+1}\,,$$

and, by (a),

$$f(n+12) \;=\; f(n+4a) \;=\; f(n)\,.$$

Whence,

$$\begin{aligned} f(a) \;&=\; f(3) \;=\; f(3+166\cdot 12) \;=\; f(1995)\,, \\ f(a+1) \;&=\; f(4) \;=\; f(4+166\cdot 12) \;=\; f(1996)\,, \\ f(a+2) \;&=\; f(5) \;=\; f(5+166\cdot 12) \;=\; f(1997)\,, \end{aligned}$$

as required.

That completes this number of the *Corner*. Send me your nice solutions and Olympiad materials!