PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 September 2001**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2601. *Proposed by Michel Bataille, Rouen, France.*

Sequences $\{u_n\}$ and $\{v_n\}$ are defined by $u_0 = 4$, $u_1 = 2$, and for all integers $n \geq 0$, $u_{n+2} = 8t^2 u_{n+1} + \left(t - \frac{1}{2}\right) u_n$, $v_n = u_{n+1} - u_n$. For which t is $\{v_n\}$ a non-constant geometric sequence?

2602*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For integers a, b and c, let $Q(a,b,c)$ be the set of all numbers $an^2 + bn + c$, where $n \in \mathbb{N} = \{0, 1, \ldots, \}$.

(a) Show that $Q(6,3,-2)$ is square-free.

(b) Determine other infinite sets $Q(a,b,c)$ with the same property.

2603. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that A, B and C are the angles of a triangle. Prove that

$$\sin A + \sin B + \sin C \leq \sqrt{\frac{15}{4} + \cos(A-B) + \cos(B-C) + \cos(C-A)}\,.$$

2604. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Determine the upper and lower bounds of $\frac{a}{a+b}+\frac{b}{b+c}-\frac{a}{a+c}$ for all positive real numbers a, b and c.

(b)* Determine the upper and lower bounds (as functions of n) of

$$\sum_{j=1}^{n-1}\frac{x_j}{x_j+x_{j+1}}-\frac{x_1}{x_1+x_n}$$

for all positive real numbers x_1, x_2, $\ldots$, x_n.

2605. *Proposed by K.R.S. Sastry, Bangalore, India.*

In triangle ABC, with median AD and internal angle bisector BE, we are given $AB=7$, $BC=18$ and $EA=ED$. Find AC.

2606. *Proposed by K.R.S. Sastry, Bangalore, India.*

A Gergonne cevian connects the vertex of a triangle to the point at which the incircle is tangent to the opposite side.

Determine the unique triangle ABC (up to similarity) in which the Gergonne cevian BE bisects the median AM, and the Gergonne cevian CF bisects the median NB.

2607. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

(a) Suppose that $q>p$ are odd primes such that $q=pn+1$, where n is an integer greater than 1. Let z be a complex number such that $z^q=1$.

Prove that $\frac{z^p-1}{z^p+1} = \sum_{j=1}^{q-1}(-1)^{\lfloor\frac{j-1}{p}\rfloor}z^j$.

(b) Suppose that $q>3$ is an odd prime such that $q=3n+2$, where n is an integer greater than 1. Let z be a complex number such that $z^q=1$.

Prove that $\frac{z^3-1}{z^3+1} = \sum_{j=1}^{q-1}(-1)^{\lfloor\frac{j-3}{3}\rfloor}z^j$.

2608*. *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that x, y, $z\geq 0$ and $x^2+y^2+z^2=1$. Prove or disprove that

(a) $1 \leq \frac{x}{1-yz}+\frac{y}{1-zx}+\frac{z}{1-xy} \leq \frac{3\sqrt{3}}{2}$;

(b) $1 \leq \frac{x}{1+yz}+\frac{y}{1+zx}+\frac{z}{1+xy} \leq \sqrt{2}$.

2609. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

A convex polygon P_n $(n \geq 4)$ has the following property:

> the $n-3$ diagonals emanating from each of the n vertices of P_n divide the corresponding angle of P_n into $n-2$ equal parts.

Determine the shape of P_n.

2610. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Let $\{F_n\}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Prove that, for $n \geq 1$,

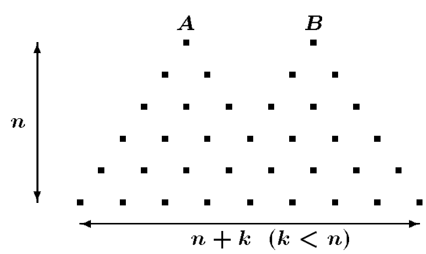
$$F_{2n} \mid (F_{3n} + (-1)^n F_n) .$$

2611. *Proposed by Michel Bataille, Rouen, France.*

Let O, H and R denote the circumcentre, the orthocentre and the circumradius of triangle ABC, and let Γ be the circle with centre O and radius $OH = \rho$. The tangents to Γ at its points of intersection with the rays $[OA)$, $[OB)$ and $[OC)$ form a triangle. Find the circumradius of this triangle as a function of R and ρ.

2612*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Two "Galton"–figures are given as follows:



(There are n levels in total; there are k levels such that there is no "intersection" between the levels emanating from A and B.)

Let two balls start at the same time from A and B. Each ball moves either ↙ or ↘ with probability $\frac{1}{2}$.

Determine the probability $P(n,k)$ $(1 \leq k < n)$ such that the two balls reach the bottom level without colliding.