THE ACADEMY CORNER

No. 38

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In this issue, we present some readers' solutions to problems from the Memorial University Undergraduate Mathematics Competition, held in March 2000. [2000 : 257]

2. Evaluate $x^3 + y^3$ where $x + y = 1$ and $x^2 + y^2 = 2$.

Solution by Catherine Shevlin, Wallsend upon Tyne, England.
Note that

$$\begin{aligned}\left(x^2+y^2\right)(x+y) &= x^3+y^3+xy(x+y)\\ &= x^3+y^3+\tfrac{1}{2}\left((x+y)^2-\left(x^2+y^2\right)\right)(x+y)\,.\end{aligned}$$

Therefore, $x^3 + y^3 = \frac{5}{2}$.

3. In triangle ABC, we have $\angle ABC = \angle ACB = 80^\circ$. P is chosen on line segment AB such that $\angle BPC = 30^\circ$. Prove that $AP = BC$.

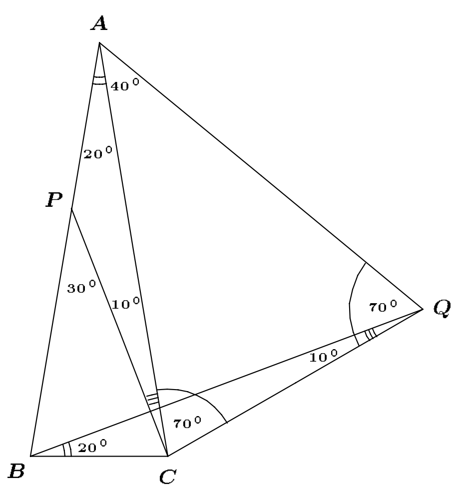
Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

Draw equilateral $\triangle ABQ$ with Q on the same side of AB as C.

Then $\triangle ACQ$ is isosceles, giving $AC = AQ$.

Since $\angle CAQ = 40^\circ$, we have $\angle ACQ = \angle AQC = 70^\circ$.

Similarly, since $\angle ABQ = 60^\circ$, we have $\angle CBQ = 20^\circ = \angle BAC$, so that $\angle ACP = \angle BPC - \angle BAC = 10^\circ$.



Thus, we have

$$\left.\begin{aligned}\angle CBQ &= \angle PAC = 20^\circ\,,\\ \angle CQB &= \angle ACP = 10^\circ\,,\\ BQ &= AC\,,\end{aligned}\right\}$$

showing that $\triangle BCQ$ is congruent to $\triangle APC$. Thus, $BC = AP$.

4. Show that $\binom{2000}{3} = (1)(1998) + (2)(1997) + \ldots + k(1999-k) + \ldots$
$\ldots + (1997)(2) + (1998)(1)$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

$$\begin{aligned}\sum_{k=1}^{1998} k(1999-k) &= 2\sum_{k=1}^{999} k(1999-k) \\ &= 2\cdot 1999 \cdot \sum_{k=1}^{999} k \;-\; 2\sum_{k=1}^{999} k^2 \,. \qquad (1)\end{aligned}$$

Also,
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \,. \qquad (2)$$

From (1) and (2), we obtain

$$\begin{aligned}\sum_{k=1}^{1998} k(1999-k) &= 2\cdot 1999 \cdot \tfrac{1}{2} \cdot 999 \cdot 1000 - \tfrac{1}{3} \cdot 999 \cdot 1000 \cdot 1999 \\ &= \tfrac{2}{3} \cdot 999 \cdot 1000 \cdot 1999 \;=\; \tfrac{1}{6} \cdot 2000 \cdot 1998 \cdot 1999 \\ &= \tbinom{2000}{3} \,.\end{aligned}$$

7. Let $f(x) = x(x-1)(x-2)\ldots(x-n)$.
   1. Show that $f'(0) = (-1)^n n!$
   2. More generally, show that if $0 \le k \le n$, then $f'(k) = (-1)^{n-k} k!(n-k)!$

Solution by Catherine Shevlin, Wallsend upon Tyne, England.

Taking logarithms and differentiating, we have

$$\begin{aligned}f'(x) &= f(x) \sum_{k=0}^{n} \frac{1}{x-k} \\ &= (x-1)(x-2)\cdots(x-n)\left(1 + \frac{x}{x-1} + \cdots + \frac{x}{x-n}\right) \,.\end{aligned}$$

Thus, $f'(0) = (-1)(-2)\cdots(-n) = (-1)^n n!$.

More generally, if $0 \le k \le n$, we have

$$\begin{aligned}f'(x) &= x(x-1)\cdots(x-k+1)(x-k-1)\cdots(x-n) \\ &\quad \times \left(\frac{x-k}{x} + \cdots + \frac{x-k}{x-k+1} + 1 + \frac{x-k}{x-k-1} + \cdots + \frac{x-k}{x-n}\right),\end{aligned}$$

showing that

$$f'(k) = (k\cdot(k-1)\cdots 2\cdot 1)\cdot(-1)\cdot(-2)\cdots(-(n-k)) = k!(-1)^{n-k}(n-k)! \,.$$

We end this issue with the problems of the 2000 William Lowell Putnam Mathematical Competition, reprinted with permission of the Mathematical Association of America.

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Problem A1.

Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, x_2, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Problem A2.

Prove that there exist infinitely many integers n such that n, $n+1$, and $n+2$ are each the sum of two squares of integers.

[*Example*: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

Problem A3.

The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5 and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

Problem A4.

Show that the improper integral

$$\lim_{B\to\infty} \int_0^B \sin(x)\sin(x^2)\,dx$$

converges.

Problem A5.

Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

Problem A6.

Let $f(x)$ be a polynomial with integer coefficients. Define a sequence a_0, a_1, $\ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

Problem B1.
Let a_j, b_j, and c_j be integers for $1 \leq j \leq N$. Assume, for each j, that at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

Problem B2.
Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$. [Here, $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ and $\gcd(m,n)$ is the greatest common divisor of m and n.]

Problem B3.
Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and $a_N \neq 0$. Let N_k denote the number of zeros (including multiplicities) of $\dfrac{d^k f}{dt^k}$.

Prove that $N_0 \leq N_1 \leq N_2 \leq \cdots$ and $\lim\limits_{k\to\infty} N_k = 2N$.

Problem B4.
Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

Problem B5.
Let S_0 be a finite set of positive integers. We define finite sets S_1, S_2, ... of positive integers as follows:

Integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n.

Show that there exist infinitely many integers N for which

$$S_N = S_0 \cup \{N + a : a \in S_0\}.$$

Problem B6.
Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space, with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

Send us your interesting solutions.

We are always pleased to receive submissions for this corner.