THE ACADEMY CORNER

No. 38

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

In this issue, we present some readers' solutions to problems from the Memorial University Undergraduate Mathematics Competition, held in March 2000. [2000 : 257]

2. Evaluate $x^3 + y^3$ where $x + y = 1$ and $x^2 + y^2 = 2$.

Solution by Catherine Shevlin, Wallsend upon Tyne, England.
Note that

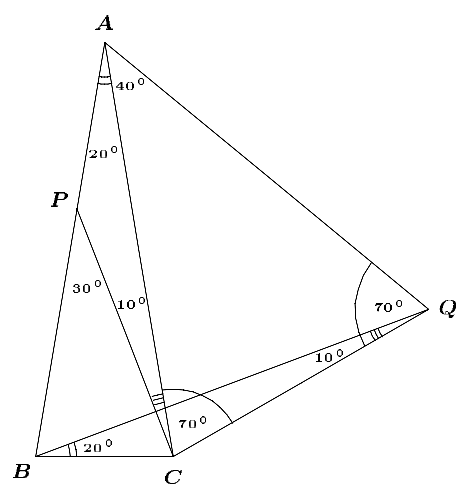
$$\begin{aligned}\left(x^2+y^2\right)(x+y) &= x^3+y^3+xy(x+y)\\ &= x^3+y^3+\tfrac{1}{2}\left((x+y)^2-\left(x^2+y^2\right)\right)(x+y)\,.\end{aligned}$$

Therefore, $x^3 + y^3 = \frac{5}{2}$.

3. In triangle ABC, we have $\angle ABC = \angle ACB = 80^\circ$. P is chosen on line segment AB such that $\angle BPC = 30^\circ$. Prove that $AP = BC$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

Draw equilateral $\triangle ABQ$ with Q on the same side of AB as C.



Then $\triangle ACQ$ is isosceles, giving $AC = AQ$.

Since $\angle CAQ = 40^\circ$, we have $\angle ACQ = \angle AQC = 70^\circ$.

Similarly, since $\angle ABQ = 60^\circ$, we have $\angle CBQ = 20^\circ = \angle BAC$, so that $\angle ACP = \angle BPC - \angle BAC = 10^\circ$.

Thus, we have

$$\left.\begin{aligned}\angle CBQ &= \angle PAC = 20^\circ\,,\\ \angle CQB &= \angle ACP = 10^\circ\,,\\ BQ &= AC\,,\end{aligned}\right\}$$

showing that $\triangle BCQ$ is congruent to $\triangle APC$. Thus, $BC = AP$.

4. Show that $\binom{2000}{3} = (1)(1998) + (2)(1997) + \ldots + k(1999-k) + \ldots$
$\ldots + (1997)(2) + (1998)(1)$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

$$\begin{aligned}\sum_{k=1}^{1998} k(1999-k) &= 2\sum_{k=1}^{999} k(1999-k) \\ &= 2\cdot 1999\cdot \sum_{k=1}^{999} k \;-\; 2\sum_{k=1}^{999} k^2\,. \qquad (1)\end{aligned}$$

Also,
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\,. \qquad (2)$$

From (1) and (2), we obtain

$$\begin{aligned}\sum_{k=1}^{1998} k(1999-k) &= 2\cdot 1999\cdot \tfrac{1}{2}\cdot 999\cdot 1000 - \tfrac{1}{3}\cdot 999\cdot 1000\cdot 1999 \\ &= \tfrac{2}{3}\cdot 999\cdot 1000\cdot 1999 \;=\; \tfrac{1}{6}\cdot 2000\cdot 1998\cdot 1999 \\ &= \tbinom{2000}{3}\,.\end{aligned}$$

7. Let $f(x) = x(x-1)(x-2)\ldots(x-n)$.
   1. Show that $f'(0) = (-1)^n n!$
   2. More generally, show that if $0 \le k \le n$, then $f'(k) = (-1)^{n-k} k!(n-k)!$

Solution by Catherine Shevlin, Wallsend upon Tyne, England.

Taking logarithms and differentiating, we have

$$\begin{aligned}f'(x) &= f(x)\sum_{k=0}^{n}\frac{1}{x-k} \\ &= (x-1)(x-2)\cdots(x-n)\left(1+\frac{x}{x-1}+\cdots+\frac{x}{x-n}\right)\,.\end{aligned}$$

Thus, $f'(0) = (-1)(-2)\cdots(-n) = (-1)^n n!$.

More generally, if $0 \le k \le n$, we have

$$\begin{aligned}f'(x) &= x(x-1)\cdots(x-k+1)(x-k-1)\cdots(x-n) \\ &\quad\times\left(\frac{x-k}{x}+\cdots+\frac{x-k}{x-k+1}+1+\frac{x-k}{x-k-1}+\cdots+\frac{x-k}{x-n}\right),\end{aligned}$$

showing that

$$f'(k) = (k\cdot(k-1)\cdots 2\cdot 1)\cdot(-1)\cdot(-2)\cdots(-(n-k)) = k!(-1)^{n-k}(n-k)!\,.$$

We end this issue with the problems of the 2000 William Lowell Putnam Mathematical Competition, reprinted with permission of the Mathematical Association of America.

SIXTY-FIRST ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION
Saturday, December 2, 2000

Problem A1.

Let A be a positive real number. What are the possible values of $\sum\limits_{j=0}^{\infty} x_j^2$, given that x_0, x_1, x_2, ... are positive numbers for which $\sum\limits_{j=0}^{\infty} x_j = A$?

Problem A2.

Prove that there exist infinitely many integers n such that n, $n+1$, and $n+2$ are each the sum of two squares of integers.

[*Example*: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

Problem A3.

The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5 and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

Problem A4.

Show that the improper integral

$$\lim_{B\to\infty} \int_0^B \sin(x)\sin(x^2)\,dx$$

converges.

Problem A5.

Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

Problem A6.

Let $f(x)$ be a polynomial with integer coefficients. Define a sequence a_0, a_1, ... of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

Problem B1.

Let a_j, b_j, and c_j be integers for $1 \leq j \leq N$. Assume, for each j, that at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

Problem B2.

Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$. [Here, $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ and $\gcd(m,n)$ is the greatest common divisor of m and n.]

Problem B3.

Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and $a_N \neq 0$. Let N_k denote the number of zeros (including multiplicities) of $\frac{d^k f}{dt^k}$.

Prove that $N_0 \leq N_1 \leq N_2 \leq \cdots$ and $\lim_{k\to\infty} N_k = 2N$.

Problem B4.

Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

Problem B5.

Let S_0 be a finite set of positive integers. We define finite sets S_1, S_2, ... of positive integers as follows:

Integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n.

Show that there exist infinitely many integers N for which

$$S_N = S_0 \cup \{N + a : a \in S_0\}.$$

Problem B6.

Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space, with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

Send us your interesting solutions.

We are always pleased to receive submissions for this corner.

THE OLYMPIAD CORNER

No. 211

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

Another year has sped past. My thanks go to our faithful readers and contributors for sending us contest materials and solutions:

Mohammed Aassila	Murray S. Klamkin
Arthur Baragar	Marcin E. Kuczma
Michel Bataille	Vedula N. Murty
Francisco Bellot Rosado	Richard Nowakowski
Andrew Blinn	Challa K.S.N. Murali Sankar
Miguel Amengual Covas	Heinz-Jürgen Seiffert
Pierre Bornsztein	Toshio Seimiya
Jacqueline Freeman	Achilleas Sinefakopoulos
J.P. Grossman	Panos E. Tsaoussoglou
Walther Janous	Edward T.H. Wang
Geoffrey A. Kandall	Sam Wong

Special thanks also go to Joanne Longworth who transforms my scribbles into a TEX file that is usually under tight time line restrictions, but always with good humour and great skill.

We begin the New Year with the problems of the Ukrainian Mathematical Olympiad, March 1997. Thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Argentina for collecting the problems for our use in the *Corner*.

UKRAINIAN MATHEMATICAL OLYMPIAD
Selected Problems
March 1997

1. (9th Grade) Cells of some rectangular board are coloured as chess board cells. In each cell an integer is written. It is known that the sum of numbers in each row is even and the sum of numbers in each column is even. Prove that the sum of all numbers in the black cells is even.

2. (10th Grade) Solve the system in real numbers

$$\begin{cases} x_1 + x_2 + \cdots + x_{1997} &= 1997 \\ x_1^4 + x_2^4 + \cdots + x_{1997}^4 &= x_1^3 + x_2^3 + \cdots + x_{1997}^3 \end{cases}$$

3. (10th Grade) Let $d(n)$ denote the greatest odd divisor of the natural number n. We define the function $f : N \to N$ as follows: $f(2n-1) = 2^n$, $f(2n) = n + \frac{2n}{d(n)}$ for all $n \in N$.
Find all k such that $f(f(\ldots f(1)\ldots)) = 1997$ where f is iterated k times.

4. (10th Grade) In space two regular pentagons $ABCDE$ and $AEKPL$ are situated so that $\angle DAK = 60°$.
Prove that the planes (ACK) and (BAL) are perpendicular.

5. (11th Grade) It is known that the equation $ax^3 + bx^2 + cx + d = 0$ with respect to x has three distinct real roots. How many roots does the equation $4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2$ have?

6. (11th Grade) Let $\mathbb{Q}^+$ denote the set of all positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$:

(a) $f(x+1) = f(x) + 1$,

(b) $f(x^2) = (f(x))^2$.

7. (11th Grade) Find the smallest n such that among any n integers there are 18 integers whose sum is divisible by 18.

8. (11th Grade) At the edges AB, BC, CD, DA of parallelepiped $ABCDA_1B_1C_1D_1$ (not necessarily right) points K, L, M, N respectively are taken. Prove that the four centres of the escribed spheres of A_1AKN, B_1BKL, C_1CLM, D_1DMN are vertices of a parallelogram.

Next we give the problems of the two papers of the Tenth Irish Mathematical Olympiad 1997. Thanks again go to Richard Nowakowski for collecting them for us when he was Canadian Team Leader at the IMO in Argentina.

TENTH IRISH MATHEMATICAL OLYMPIAD
May 10, 1997
First Paper

1. Find (with proof) all pairs of integers (x, y) satisfying the equation

$$1 + 1996x + 1998y = xy .$$

2. Let ABC be an equilateral triangle. For a point M inside ABC, let D, E, F be the feet of the perpendiculars from M onto BC, CA, AB, respectively. Find the locus of all such points M for which $\angle FDE$ is a right angle.

3. Find all polynomials $p(x)$ satisfying the equation

$$(x-16)p(2x) = 16(x-1)p(x)$$

for all x.

4. Let a, b, c be non-negative real numbers. Suppose that $a+b+c \geq abc$. Prove that $a^2+b^2+c^2 \geq abc$.

5. Let S be the set of all odd integers greater than one. For each $x \in S$, denote by $\delta(x)$ the unique integer satisfying the inequality

$$2^{\delta(x)} \;<\; x \;<\; 2^{\delta(x)+1}\,.$$

For a, $b \in S$, define

$$a * b \;=\; 2^{\delta(a)-1}(b-3)+a\,.$$

[For example, to calculate $5*7$ note that $2^2 < 5 < 2^3$, so that $\delta(5)=2$, and hence, $5*7 = 2^{2-1}(7-3)+5 = 13$. Also $2^2 < 7 < 2^3$, so that $\delta(7)=2$ and $7*5 = 2^{2-1}(5-3)+7 = 11$].

Prove that if a, b, $c \in S$, then

(a) $a*b \in S$ and

(b) $(a*b)*c \;=\; a*(b*c)$.

6. Given a positive integer n, denote by $\sigma(n)$ the sum of all the positive integers which divide n. [For example, $\sigma(3) = 1+3 = 4$, $\sigma(6) = 1+2+3+6 = 12$, $\sigma(12) = 1+2+3+4+6+12 = 28$].

We say that n is abundant if $\sigma(n) > 2n$. (Thus, for example, 12 is abundant). Let a, b be positive integers and suppose that a is abundant. Prove that ab is abundant.

7. $ABCD$ is a quadrilateral which is circumscribed about a circle Γ (that is, each side of the quadrilateral is tangent to Γ). If $\angle A = \angle B = 120^\circ$, $\angle D = 90^\circ$ and BC has length 1, find, with proof, the length of AD.

8. Let A be a subset of $\{0, 1, 2, 3, \ldots, 1997\}$ containing more than 1000 elements. Prove that either A contains a power of 2 (that is, a number of the form 2^k with k a non-negative integer) or there exist two distinct elements $a, b \in A$ such that $a+b$ is a power of 2.

9. Let S be the set of all natural numbers n satisfying the following conditions:

(a) n has 1000 digits

(b) all the digits of n are odd and

(c) the absolute value of the difference between adjacent digits of n is 2.

Determine the number of distinct elements of S.

10. Let p be a prime number and n a natural number and let $T = \{1, 2, 3, \ldots, n\}$. Then n is called *p-partitionable* if there exist non-empty subsets $T_1, T_2, \ldots, T_p$ of T such that

(i) $T = T_1 \cup T_2 \cup \cdots \cup T_p$

(ii) $T_1, T_2, \ldots, T_p$ are disjoint (that is $T_i \cap T_j$ is the empty set for all i, j with $i \neq j$) and

(iii) the sum of the elements in T_i is the same for $i = 1, 2, \ldots, p$.

[For example, 5 is 3-partitionable since, if we take $T_1 = \{1, 4\}$, $T_2 = \{2, 3\}$, $T_3 = \{5\}$, then (i), (ii) and (iii) are satisfied. Also 6 is 3-partitionable since, if we take $T_1 = \{1, 6\}$, $T_2 = \{2, 5\}$, $T_3 = \{3, 4\}$, then (i), (ii) and (iii) are satisfied].

(a) Suppose that n is p-partitionable. Prove that p divides n or $n + 1$.

(b) Suppose that n is divisible by $2p$. Prove that n is p-partitionable.

As a third contest this number, we give the problems of the Hungary-Israel Bi-national Mathematics Competition, 1997. Thanks go to Richard Nowakowski for collecting the problems for use in the *Corner* when he was Team Leader to the IMO in Argentina.

THE HUNGARY-ISRAEL BI-NATIONAL MATHEMATICAL COMPETITION 1997

April 13–20, 1997

Time: 4.5 hours

1. Is there an integer N such that

$$(\sqrt{1997} - \sqrt{1996})^{1998} = \sqrt{N} - \sqrt{N-1}\,?$$

2. Find all real numbers α with the following property: for any positive integer n there exists an integer m such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{3n}\,.$$

3. ABC is an acute angled triangle whose circumcentre is O. The intersection points of the diameters of the circumcircle, passing through A, B, C, with the opposite sides are A_1, B_1, C_1 respectively. The circumradius of the triangle ABC is of length $2p$ where p is a prime. The lengths OA_1, OB_1, OC_1 are integers. What are the lengths of the sides of the triangle?

4. What is the number of distinct sequences of length 1997 that can be formed by using the letters A, B, C, where each letter appears an odd number of times?

5. The three squares ACC_1A'', ABB_1A', $BCDE$ are constructed on the sides of a given triangle ABC, outwards. The centre of the square $BCDE$ is P. Prove that the three lines $A'C$, $A''B$ and PA pass through one point.

6. Can a closed disk be decomposed into a union of two congruent parts having no common points?

As a fourth contest to resharpen your problem solving skills we give the problems of the 10^{th} Grade, 36^{th} Armenian National Olympiad in Mathematics. Thanks go to Richard Nowakowski for collecting the problems for use in the *Corner* when he was Team Leader to the IMO in Argentina.

36^{th} ARMENIAN NATIONAL OLYMPIAD IN MATHEMATICS

Republic of Armenia, 1997

10^{th} Grade

1. Let

$$p(x) \;=\; (x-a_1)^{n_1}(x-a_1)^{n_2}(x-a_3)^{n_3}$$

be a polynomial, such that

$$p(x)-1 \;=\; (x-b_1)^{k_1}(x-b_2)^{k_2}(x-b_3)^{k_3}\,,$$

where the numbers a_1, a_2, a_3, as well as b_1, b_2, b_3, are distinct and n_1, n_2, n_3, k_1, k_2, k_3 are natural numbers. Prove, that the degree of the polynomial $p(x)$ does not exceed 5.

2. Suppose a and b are natural numbers, such that $(a+b)$ is an odd number. Prove that for any division of the set of natural numbers into two groups, there will be two numbers from the same group, the difference of which is either a or b.

3. Prove that, for any points A, B, C, D, E, F, the following inequality holds:

$$AD^2+BE^2+CF^2 \;\le\; 2(AB^2+BC^2+CD^2+DE^2+EF^2+FA^2)\,.$$

4. It is known that the function $f(x)$ is defined on the set of natural numbers, taking values from the natural numbers, and that it satisfies the following conditions:

(a) $f(xy)=f(x)+f(y)-1$ for any x, $y\in N$,

(b) the equality $f(x)=1$ is true for finitely many numbers,

(c) $f(30)=4$.

Find $f(14400)$.

5. The rectangle P_1 is inscribed in the rectangle P_2 which has sides c, d ($c \leq d$), where d is less than the long side of P_1. Prove that, for any angle α between two lines, containing any two sides of rectangles P_1 and P_2, the following inequality holds:

$$\sin 2\alpha \;\geq\; \frac{c}{d}\,.$$

Now we turn to solutions by our readers to problems posed in the *Corner*. First a nice alternate solution to a problem treated in 1999.

3. [1998 : 69; 1999 : 201]

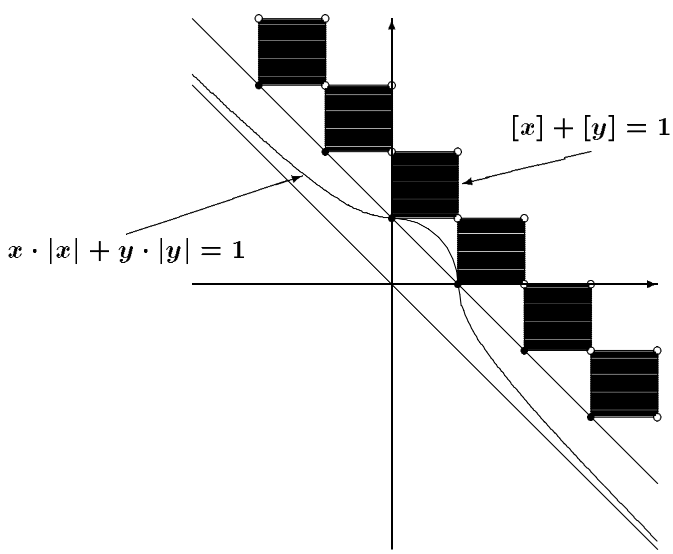
Solve the following system of equations:

$$x \cdot |x| + y \cdot |y| \;=\; 1, \quad [x] + [y] \;=\; 1\,,$$

in which $|t|$ and $[t]$ represent the absolute value and the integer part of the real number t.

Alternate Solution by Michel Bataille, Rouen, France.

It has been shown that the only solutions are $(x, y) = (1, 0)$ and $(0, 1)$ [1999 : 201]. Here is an alternative proof 'without words' ...



Now we turn to reader solutions to problems of the April 1999 *Corner*. The first group of solutions are to problems of the 47$^{\text{th}}$ Polish Mathematical Olympiad [1999 : 132–133].

1. Find all pairs (n, r), with n a positive integer, r a real number, for which the polynomial $(x+1)^n - r$ is divisible by $2x^2 + 2x + 1$.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Bataille (although they were all very similar).

By division, we can find a polynomial $q(x)$ and two real numbers a, b such that:

$$(x+1)^n - r \;=\; (2x^2 + 2x + 1) \cdot q(x) + ax + b\,.$$

Since the complex roots of $2x^2+2x+1$ are $-\frac{1}{2}+\frac{i}{2}$ and $-\frac{1}{2}-\frac{i}{2}$, the numbers a, b satisfy the system:

$$\begin{cases} a\left(-\frac{1}{2}+\frac{i}{2}\right)+b &= \left(1+\frac{-1+i}{2}\right)^n - r \\ a\left(-\frac{1}{2}-\frac{i}{2}\right)+b &= \left(1+\frac{-1-i}{2}\right)^n - r\,. \end{cases}$$

Taking into account:

$$1+\frac{-1+i}{2} \;=\; \frac{1+i}{2} \;=\; \frac{1}{\sqrt{2}}e^{i\pi/4}$$

and

$$1+\frac{-1-i}{2} \;=\; \frac{1-i}{2} \;=\; \frac{1}{\sqrt{2}}e^{-i\pi/4}\,,$$

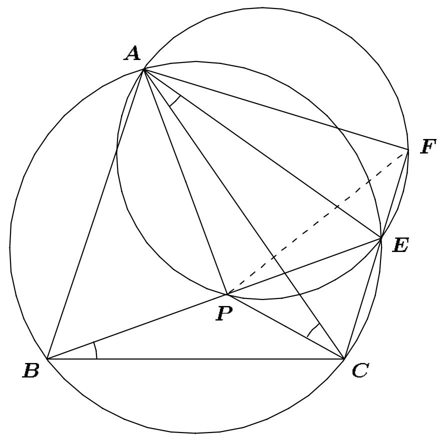
we get, by subtraction:

$$ai \;=\; 2^{-n/2}(e^{in\pi/4} - e^{-in\pi/4}) \;=\; 2^{-n/2}\cdot 2i\sin(n\pi/4)\,.$$

Hence, $a = 0$ if and only if $\sin(n\pi/4) = 0$; that is, n is a multiple of 4. Assuming that $n = 4k$, where k is a positive integer, we obtain that $b = 2^{-2k}(e^{i\pi})^k - r$, so that $b = 0$ if and only if $r = \frac{(-1)^k}{4^k}$. Thus, the solutions are the pairs $\left(4k, \frac{(-1)^k}{4^k}\right)$ where k is a positive integer.

2. Given is a triangle ABC and a point P inside it satisfying the conditions: $\angle PBC = \angle PCA < \angle PAB$. Line BP cuts the circumcircle of ABC at B and E. The circumcircle of triangle APE meets line CE at E and F. Show that the points A, P, E, F are consecutive vertices of a quadrilateral. Also show that the ratio of the area of quadrilateral $APEF$ to the area of triangle ABP does not depend on the choice of P.

Solution by Toshio Seimiya, Kawasaki, Japan.



Since

$$\begin{aligned} \angle PFC &= \angle PAE = \angle PAC + \angle EAC = \angle PAC + \angle EBC \\ &= \angle PAC + \angle PBC \\ &< \angle PAC + \angle PAB = \angle BAC = \angle BEC = \angle PEC\,. \end{aligned}$$

Thus, F is a point on CE beyond E.

Therefore, A, P, E, F are consecutive vertices of a quadrilateral. Since $\angle EAC = \angle EBC = \angle PBC = \angle PCA$, we have $PC \| AE$. Thus, we have $[APE] = [ACE]$, where $[A_1A_2 \ldots A_n]$ denotes the area of n-gon A_1, A_2, $\ldots$, A_n.

Hence,

$$[APEF] = [APE] + [AEF] = [ACE] + [AEF] = [ACF]\,.$$

Since

$$\angle ACF = \angle ACE = \angle ABE = \angle ABP\,,$$

and

$$\angle AFC = \angle AFE = \angle APB\,,$$

we have $\triangle ACF \sim \triangle ABP$.

Thus, we get $[ACF] : [ABP] = AC^2 : AB^2$; that is

$$[APEF] : [ABP] = AC^2 : AB^2 \quad (= \text{constant}).$$

Therefore, $[APEF] : [ABP]$ does not depend on the choice of P.

3. Let $n \geq 2$ be a fixed natural number and let $a_1, a_2, \ldots, a_n$ be positive numbers whose sum equals 1.

(a) Prove the inequality

$$2\sum_{i<j} x_i x_j \leq \frac{n-2}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i}$$

for any positive numbers $x_1, x_2, \ldots, x_n$ summing to 1.

(b) Determine all n-tuples of positive numbers $x_1, x_2, \ldots, x_n$ summing to 1 for which equality holds.

Solutions by Mohammed Aassila, Strasbourg, France; by Pierre Bornsztein, Courdimanche, France; by George Evagelopoulos, Athens, Greece; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Klamkin's solution.

(a) Replacing a_i in the numerator by $(a_i - 1) + 1$ and simplifying, we get

$$1 \leq (n-1)\sum \frac{x_i^2}{1-a_i} = \left(\sum (1-a_i)\right)\left(\sum \frac{x_i^2}{1-a_i}\right),$$

where the sums here and subsequently are over $i = 1$ to n. By Cauchy's inequality, the right hand side is $\geq (\sum x_i)^2 = 1$.

(b) There is equality if and only if $x_i = k(1 - a_i)$ where $k = \frac{1}{n-1}$.

4. Let $ABCD$ be a tetrahedron with

$$\angle BAC = \angle ACD \quad \text{and} \quad \angle ABD = \angle BDC\,.$$

Show that edges AB and CD have equal lengths.

Solutions by Michel Bataille, Rouen, France; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Toshio Seimiya, Kawasaki, Japan. We give Klamkin's solution and remarks.

It is to be noted that there is a tacit assumption the figure is 3-dimensional. The result is not valid if $ABCD$ is a trapezoid with AB parallel to CD.

Letting $b = AB$, $c = AC$, $d = AD$, $b_1 = CD$, $c_1 = BD$, and $d_1 = BC$, and applying the Law of Cosines, we get

$$2c\cos\angle BAC = \frac{b^2+c^2-d_1^2}{b} = \frac{b_1^2+c^2-d^2}{b_1}\,, \qquad (1)$$

$$2c_1\cos\angle ABD = \frac{b_1^2+c_1^2-d_1^2}{b_1} = \frac{b^2+c_1^2-d^2}{b}\,. \qquad (2)$$

Equations (1) and (2) reduce to

$$(bb_1 - c^2)(b - b_1) = b_1 d_1^2 - bd^2\,, \qquad (1')$$

$$(bb_1 - c_1^2)(b - b_1) \;=\; b_1 d^2 - b d_1^2\,. \tag{2'}$$

Adding the latter two equations, we get

$$(b - b_1)(2bb_1 + d^2 + d_1^2 - c^2 - c_1^2) \;=\; 0\,. \tag{3}$$

We now show that the second factor of (3) is > 0, so that $b = b_1$. Letting $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ denote the vectors AB, AC, and AD, respectively, the second factor is also given by

$$2|\mathbf{B}||\mathbf{C}-\mathbf{D}| + \mathbf{D}^2 + (\mathbf{B}-\mathbf{C})^2 - \mathbf{C}^2 - (\mathbf{B}-\mathbf{D})^2 \;=\; 2(|\mathbf{B}||\mathbf{C}-\mathbf{D}| - \mathbf{B}(\mathbf{C}-\mathbf{D}))\,.$$

Hence, this factor is > 0 provided AB is not parallel to CD. It now also follows from (1) or (2) that $d = d_1$.

Comment. A related problem in which the angle hypotheses were

$$\angle BAD \;=\; \angle BCD \quad \text{and} \quad \angle ABC \;=\; \angle ADC$$

and the conclusions the same, occurred as problem B-5 in the 1972 William Lowell Putnam Competition. This result is due to Thebault [1] and there is a simpler proof by Bottema [2].

References:

[1] V. Thebault, On the skew quadrilateral, *Amer. Math. Monthly* **60** (1953) 102–105.

[2] O. Bottema, Notes on the skew quadrilateral, *Amer. Math. Monthly* **61** (1954) 692–693.

5. For a natural number $k \geq 1$ let $p(k)$ denote the least prime number which is not a divisor of k. If $p(k) > 2$, define $q(k)$ to be the product of all primes less than $p(k)$, and if $p(k) = 2$, set $q(k) = 1$. Consider the sequence

$$x_0 \;=\; 1\,, \quad x_{n+1} \;=\; \frac{x_n p(x_n)}{q(x_n)} \quad \text{for} \quad n \;=\; 0, 1, 2, \ldots\,.$$

Determine all natural numbers n such that $x_n = 111\,111$.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Bornsztein's solution.

We will prove that $x_n = 111\,111$ if and only if $n = 2106$. We adopt the convention $\prod_{x\in\phi} x = 1$ and $p_1 = 2$, p_i denotes the i^{th} prime (in increasing order).

Then, for every $k \in \mathbb{N}^*$, $q(x) = \prod_{p_i < p(k)} p_i$. We have $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 2 \times 3$, $x_4 = 5$, $x_5 = 2 \times 5$, $x_6 = 3 \times 5$, $x_7 = 2 \times 3 \times 5$, $x_8 = 7$.

By induction on $n \geq 1$, we prove that, if

$$n \;=\; \sum_{i \geq 0} \alpha_i 2^i, \quad \alpha_i \in \{0, 1\} \quad \text{(binary expansion of } n\text{)}\,,$$

then

$$x_n = \prod_{i \geq 0} p_{i+1}^{\alpha_i} . \tag{1}$$

For $n = 1$, $\alpha_0 = 1$ and $\alpha_i = 0$ for $i \geq 1$, then

$$\prod_{i \geq 0} p_{i+1}^{\alpha_i} = 2 = x_1 ,$$

and so, (1) holds for $n = 1$.

Let $n \geq 1$ be a fixed integer. We suppose (1) holds for n,

$$n = \sum_{i \geq 0} \alpha_i 2^i .$$

Let i_0 be the smallest integer such that $\alpha_{i_0} = 0$. Then $n + 1 = \sum_{i \geq 0} \beta_i 2^i$ where:

$$\begin{cases} \beta_i = 0 & \text{if } i < i_0 , \\ \beta_{i_0} = 1 , & \\ \beta_i = \alpha_i & \text{if } i > i_0 . \end{cases} \tag{2}$$

From the induction hypothesis (1) we have:

$$x_n = \prod_{i \geq 0} p_{i+1}^{\alpha_i} = \prod_{i < i_0} p_{i+1} \times \prod_{i > i_0} p_{i+1}^{\alpha_i} \quad (\text{since } \alpha_{i_0} = 0 \text{ and } \alpha_i = 1 \text{ for } i < i_0).$$

Then $p(x_n) = p_{i_0+1}$ and $q(x_n) = \prod_{i<i_0} p_{i+1}$.

It follows that

$$\begin{aligned} x_{n+1} &= \frac{\prod_{i<i_0} p_{i+1} \times \prod_{i>i_0} p_{i+1}^{\alpha_i} \times p_{i_0+1}}{\prod_{i<i_0} p_{i+1}} = p_{i_0+1} \prod_{i>i_0} p_{i+1}^{\alpha_i} \\ &= \prod_{i<i_0} p_{i+1}^{\beta_i} \times p_{i_0+1}^{\beta_{i_0}} \times \prod_{i>i_0} p_{i+1}^{\beta_i} \quad (\text{from (2)}) \\ &= \prod_{i \geq 0} p_{i+1}^{\beta_i} . \end{aligned}$$

Then (1) holds for $n + 1$, and the induction is achieved.

Since $x_0 = 1$, we have $x_0 \neq 111\,111$.

For $n \geq 1$, from (1), we deduce:

$$\begin{aligned} x_n = 111\,111 &\iff x_n = 3 \times 7 \times 11 \times 13 \times 37 \\ &\qquad\;\; = p_2 p_4 p_5 p_6 p_{12} , \\ &\iff n = 2^{11} + 2^5 + 2^4 + 2^3 + 2^1 , \\ &\iff n = 2106 , \quad \text{as claimed.} \end{aligned}$$

Next we turn to solutions from our readers to problems of the 10^{th} Nordic Mathematical Contest [1999 : 133–134].

1. Prove the existence of a positive integer divisible by 1996 the sum of whose decimal digits is 1996.

Solutions by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Bataille.

Let $S(n)$ denote the sum of the decimal digits of the positive integer n. We observe:

$$S(1996) \;=\; 25 \quad \text{and} \quad S(2\times 1996) \;=\; S(3992) \;=\; 23\,.$$

Since $1996 = 78 \times 25 + 23 + 23$, the integer

$$n = 19961996\ldots 199639923992$$

(where the group of digits 1996 is repeated 78 times) satisfies $S(n) = 1996$. Moreover, $n = 3992 + 3992 \times 10^4 + 1996 \times 10^8 + \cdots + 1996 \times 10^{4\times 79}$ is clearly a multiple of 1996.

2. Determine all real x such that

$$x^n + x^{-n}$$

is an integer for any integer n.

Solutions by Mohammed Aassila, Strasbourg, France; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Bataille's write-up.

Let x be a non-zero real number and, for any integer n, let $a_n = x^n + x^{-n}$. Since $a_{-n} = a_n$, we can consider only non-negative integers n.

We have

$$a_0 \;=\; 2\,;\, a_1 \;=\; x + \frac{1}{x}\,; \quad \ldots\,;$$

$$a_{n+1} \;=\; x^{n+1} + \frac{1}{x^{n+1}} \;=\; \left(x^n + \frac{1}{x^n}\right)\left(x + \frac{1}{x}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right)\,;$$

so that, for $n \geq 1$, we have $a_{n+1} = a_n a_1 - a_{n-1}$.

With the help of this relation, an easy induction shows that a_n is an integer for all positive integers n if (and only if) a_1 is an integer.

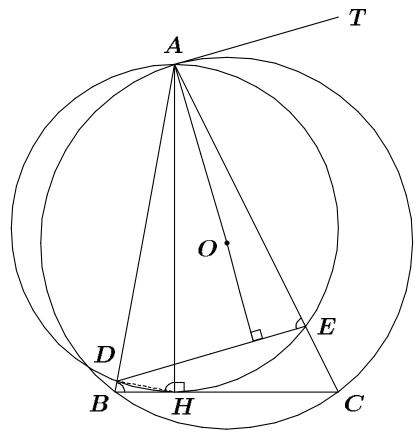
Now, the equation $x + \frac{1}{x} = k$, where k is an integer, is equivalent to $x^2 - kx + 1 = 0$. Hence, the suitable x are the real numbers

$$\frac{k + \sqrt{k^2 - 4}}{2} \quad \text{and} \quad \frac{k - \sqrt{k^2 - 4}}{2}\,,$$

where k is any integer such that $|k| \geq 2$.

3. A circle has the altitude from A in a triangle ABC as a diameter, and intersects AB and AC in the points D and E, respectively, different from A. Prove that the circumcentre of triangle ABC lies on the altitude from A in triangle ADE, or it produced.

Solutions by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.



Let H be the foot of the altitude from A to BC. Then

$$AH \perp BC\,.$$

Since AH is the diameter of the circle $ADHE$, we have

$$\angle ADH \;=\; 90^\circ\,.$$

Since $\angle AHB = 90^\circ$ and $HD \perp AB$, we get

$$\angle AHD \;=\; \angle ABH \;=\; \angle ABC\,.$$

Since $\angle AHD = \angle AED$ (because A, D, H, E are concyclic), we have

$$\angle ABC \;=\; \angle AED\,. \qquad (1)$$

Let O be the circumcentre of $\triangle ABC$, and let AT be the tangent at A to the circumcircle of $\triangle ABC$. Then

$$\angle TAC \;=\; \angle ABC\,. \qquad (2)$$

From (1) and (2) we get $\angle AED \;=\; \angle TAC$. Thus, $DE \perp AT$. Since $AT \perp AO$, we have $AO \perp DE$.

Therefore, the perpendicular from A to DE passes through O.

4. A real-valued function f is defined for positive integers, and a positive integer a satisfies

$$f(a) \;=\; f(1995)\,, \quad f(a+1) \;=\; f(1996)\,, \quad f(a+2) \;=\; f(1997)\,,$$

$$f(n+a) \;=\; \frac{f(n)-1}{f(n)+1} \quad \text{for any positive integer } n\,.$$

(a) Prove that $f(n+4a) = f(n)$ for any positive integer n.

(b) Determine the smallest possible value of a.

Solutions by Mohammed Aassila, Strasbourg, France; and by Pierre Bornsztein, Courdimanche, France. We give Aassila's write-up.

(a) From $f(n+a) = \frac{f(n)-1}{f(n)+1}$, we deduce that

$$f(n+2a) \;=\; f((n+a)+a) = \frac{\frac{f(n)-1}{f(n)+1} - 1}{\frac{f(n)-1}{f(n)+1} + 1} \;=\; -\frac{1}{f(n)} \qquad \text{and}$$

$$f(n+4a) \;=\; f((n+2a)+2a) \;=\; -\frac{1}{f(n+2a)} \;=\; f(n)\,.$$

(b) The smallest possible value of a is 3. Indeed, if $a = 1$, then

$$f(1) = f(a) = f(1995) = f(3+498\cdot 4a) = f(3) = f(1+2a) = -\frac{1}{f(1)}\,,$$

and hence,

$$(f(1))^2 \;=\; -1 \;, \quad \text{which is impossible.}$$

If $a = 2$, then

$$\begin{aligned} f(2) \;&=\; f(a) \;=\; f(1995) \;=\; f(3+249\cdot 4a) \;=\; f(3) \;=\; f(a+1) \\ &=\; f(1996) \;=\; f(4+249\cdot 4a) \;=\; f(4) \;=\; f(2+a) \;=\; \frac{f(2)-1}{f(2)+1}\,, \end{aligned}$$

and hence, $(f(2))^2 = -1$, which is impossible.

If $a = 3$, we should be sure that $f(n) \neq -1$ for all $n \in \mathbb{N}$. Let $f(1)$, $f(2)$ and $f(3)$ be chosen arbitrarily different from -1, 0, 1. Then, for $n = 1$, 2 or 3, and since $f(n) \neq -1, 0, 1$, none of $f(n)$, $f(n+3) = \frac{f(n)-1}{f(n)+1}$, $f(n+6) = -\frac{1}{f(n)}$ and $f(n+9) = -\frac{f(n)+1}{f(n)-1}$ is equal to -1, and then, by (a), $f(n) \neq -1$ for all $n \in \mathbb{N}$.

By construction, we have

$$f(n+a) \;=\; f(n+3) \;=\; \frac{f(n)-1}{f(n)+1}\,,$$

and, by (a),

$$f(n+12) \;=\; f(n+4a) \;=\; f(n)\,.$$

Whence,

$$\begin{aligned} f(a) \;&=\; f(3) \;=\; f(3+166\cdot 12) \;=\; f(1995)\,, \\ f(a+1) \;&=\; f(4) \;=\; f(4+166\cdot 12) \;=\; f(1996)\,, \\ f(a+2) \;&=\; f(5) \;=\; f(5+166\cdot 12) \;=\; f(1997)\,, \end{aligned}$$

as required.

That completes this number of the *Corner*. Send me your nice solutions and Olympiad materials!

BOOK REVIEWS

ALAN LAW

Confronting the Core Curriculum, by John A. Dossey, published by the Mathematical Association of America, 1998,
ISBN 0-88385-155-5, softcover, 136+ pages, $38.50 (U.S.).
Reviewed by ***Ron Scoins****, University of Waterloo, Waterloo, Ontario.*

Subtitled *Considering Change in the Undergraduate Mathematics Major*, this book is a summary of the proceedings of a 1994 conference held at West Point. The conference was organized as a response to the need for changes to core curriculum in view of the four-fold increase in the number of students studying mathematics and the realization that core mathematics must serve the needs of all students, not just math majors.

The book starts by reviewing the efforts over the past 40 years of the MAA Committee on the Undergraduate Program in Mathematics to address the goals and content of core curriculum. In spite of many excellent recommendations on ways to modify core to serve the needs of all students in the mathematical sciences, it is disappointing to realize that not many math departments have made an effort to meet this challenge. The point is also made that content and pedagogical practices should be considered simultaneously. Suggestions are given on how to establish a student growth model to ensure a successful program.

The next section describes a bold and innovative curriculum coupled with student and faculty growth models that have been in place since 1990. The "7 into 4" core curriculum is an integrated program that includes topics from discrete, continuous, linear, non-linear, deterministic, and stochastic mathematics. It has gone through several iterations since its introduction. For departments considering core curriculum reform, it is a good framework from which to start.

Section II of the book reports on the central agenda of the conference. It outlines a core curriculum designed as if most students will not take the next course in that branch but without cramming "everything" into it. Proposals are put forward for the inclusion of Discrete Mathematics, Calculus, Linear Algebra, Differential Equations, and Probability and Statistics to be the basis of a core mathematics program. The rationale for inclusion, along with suggested topics, and pedagogical practices to enhance learning for each of these five areas, was presented by a prominent mathematician from that discipline. Responses to these proposals are also reported. The responses are very supportive of the ideas put forth by the presenters.

I was disappointed that the inclusion of Computer Science was not considered to be an essential part of core mathematics. (This may have something to do with the culture at Waterloo.)

As a response to the recommendations made at the 1994 conference, a follow-up workshop was held in 1995 at which four universities reported on revisions to their core curriculum. A summary of this workshop is also included in the book.

Most educational jurisdictions in North America (including Canada) have revised the mathematics curriculum for elementary and secondary schools within the past five years. This, along with the challenge of educating a broader base of university students for participation in a technological society, suggests that university mathematics departments need to review and possibly revise their core mathematics curricula. I strongly recommend reading this book as a first step in that process.

Magic Tricks, Card Shuffling and Dynamic Computer Memories,
by S. Brent Morris,
published by the Mathematical Association of America, 1998,
ISBN 0-88385-527-5, softcover, xviii + 148 pages, $28.95 (U.S.).
Reviewed by **Paul J. Schellenberg**, *University of Waterloo, Waterloo, Ontario.*

This book may appeal to readers on several different levels — the mystery and delight of a well-performed magic trick, the magic explained, mastering the perfect shuffle, and the mathematics of perfect shuffles.

The perfect shuffle requires the dealer to divide a deck of $2n$ cards precisely in half, and then perfectly interlace the cards of the two halves. This perfect shuffle can be performed in two ways, an *out-shuffle* which leaves the top and bottom cards on top and bottom, respectively, and an *in-shuffle* which does not. For example, a deck consisting of the cards $1, 2, 3, 4, 5, 6, 7, 8$, in that order, will have the order $1, 5, 2, 6, 3, 7, 4, 8$ after an out-shuffle and the order $5, 1, 6, 2, 7, 3, 8, 4$ after an in-shuffle. The definition of a perfect shuffle is generalized to decks with an odd number of cards. These perfect shuffles permute the cards of a deck in such a precise fashion that one can mathematically determine the location of any card after one or a series of such shuffles. For example, after only 8 out-shuffles of a deck of 52 cards, the deck is restored to its original order!

The author describes how the properties of the perfect shuffle are exploited to accomplish some delightful magic tricks. The book has five chapters, each beginning with a spectator's description of a magic trick using a deck of cards. Morris then explains some mathematical properties of perfect shuffles and finally concludes the chapter by revealing how to perform the trick. In the fifth chapter, Morris applies the mathematics of perfect shuffles to describe how data can be retrieved efficiently from a volatile dynamic computer memory — an application which is primarily of theoretical interest, as developments in computer memory design have eliminated the need for dynamic memories.

The author has included a substantial bibliography to the literature of perfect shuffles, and describes some of their history. There are also four appendices. The fourth one, entitled "A Lagniappe," describes three further magic tricks, two of which rely on special properties described in this book.

The combinatorial mathematics of perfect shuffles begins at a fairly elementary level and becomes increasingly technical as it proceeds to its ultimate application to dynamic computer memories. The reader can enter into these mathematical developments as deeply as he or she chooses. The description of the magic tricks is not impeded by the complexities of the mathematics.

Among the general population, there are those who have a natural bent toward entertaining and performing. Many of you will find this book a treasure trove of material to delight and astonish your listeners.

Correction

In Walther Janous' letter [2000 : 467], there was a confusion in the dates of the references. Here is a corrected version:

References.

[1] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford, New York 1965.

[2] A. Makowski, Problem 3932, *Mathesis* 69 (1960), 65.

[3] D.S. Mitrinović, J. Sándor and B. Crstici, *Handbook of Number Theory*, Dordrecht, Boston, London, 1996.

A Heron Difference

K.R.S. Sastry

Is it possible that the lengths of two sides of a primitive Heron triangle have a common factor? The triangle with sides 9, 65, 70 and area 252 shows that this is possible. But notice that the common factor 5 is a prime of the form $4\lambda+1$. Surprisingly, however, such a common factor cannot be a prime of the form $4\lambda - 1$.

Heron's name is familiar to anyone who has used the formula

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\,, \qquad s = \tfrac{1}{2}(a+b+c)$$

to calculate the area Δ of a triangle with sides a, b, c. In a **Heron** triangle a, b, c, and Δ are positive integers. A Heron triangle is called **primitive** when $\gcd(a,b,c) = 1$. If a Heron triangle contains a right angle, then it is called a **Pythagorean** triangle. It is easy to see that there is no primitive Pythagorean triangle, where two sides have a common factor. The situation for Heron triangles is different. The aim of this paper is to prove that if there is a common (prime) factor of two sides of a primitive Heron triangle, then it is always a prime of the form $4\lambda + 1$.

Definitions and Observations

A Pythagorean triangle is a right triangle with integer sides such as $(3,4,5)$ or $(10,24,26)$. A **rational** Pythagorean triangle is a right triangle with rational side lengths such as $(\frac{3}{2}, 2, \frac{5}{2})$ or with integer side lengths. A primitive Pythagorean triangle has the $\gcd$ of its side lengths equal to 1. It is well known, see [1, 3, 6], that all primitive Pythagorean triangles have sides of length:

$$m^2 - n^2\,,\ 2mn\,,\ m^2 + n^2\,, \tag{1}$$

with the pair m, n being relatively prime integers of opposite parity. All Pythagorean triangles then have sides of length

$$k(m^2 - n^2)\,,\ k(2mn)\,,\ k(m^2 + n^2)\,, \qquad k = 1, 2, 3, \ldots. \tag{2}$$

A **rational** Heron triangle is defined analogously to its Pythagorean counterpart. A Pythagorean triangle is a Heron triangle but a Heron triangle may not be Pythagorean. Any Heron triangle can be split into an adjoin of two rational Pythagorean triangles. This is because at least one altitude, certainly the one to the longest side, must lie within the triangle. The diagrams in Figure 1 show such a splitting of the $(9,65,70)$ Heron triangle.

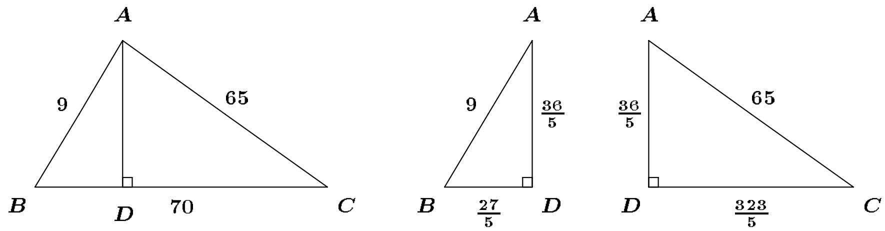


Figure 1.

Notice that these rational Pythagorean triangles ABD, ADC can be made Pythagorean ones by enlarging their sides, scale factor 5 in this case. See Figure 2.

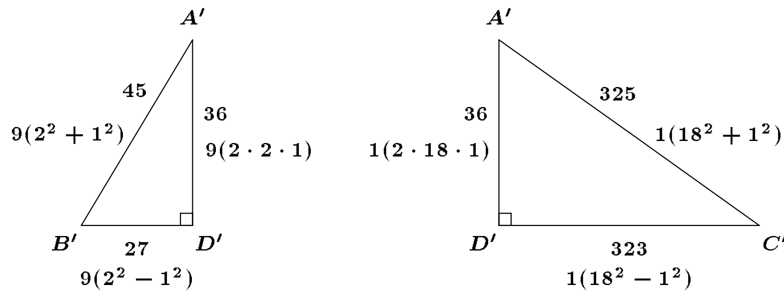


Figure 2.

Also notice that the sides of these Pythagorean triangles have the form $k_i(m_i^2 - n_i^2)$, $k_i(2m_in_i)$, $k_i(m_i^2 + n_i^2)$, $i = 1, 2$ as described in (2). One can also observe the reverse phenomenon to arrive at the Heron triangle $(45, 325, 27 + 323 = 350)$ and then, by disregarding the $\gcd 5$ of the sides, obtain the primitive Heron triangle $(9, 65, 70)$. In the above example it so happened that the common side has the same form $k_1(2m_1n_1)$ and $k_2(2m_2n_2)$ in both Pythagorean triangles. It is quite possible that the common side has the form $k_1(2m_1n_1)$ in one component and the form $k_2(m_2^2 - n_2^2)$ in the other component Pythagorean triangle. To see this look at the Heron triangle $(25, 52, 63)$ with its altitude drawn to the side 63.

Number Theory Results

To establish our Theorem we need the following well-known results available in a number theory textbook containing a discussion on integers that are sums of two integer squares, [2, 5] for example or see [1, 3, 4].

> A prime of the form $4\lambda + 1$ is uniquely a sum of two relatively prime integer squares. A divisor of a sum of two relatively prime integer squares is itself such a sum.
> A prime of the form $4\lambda - 1$ is not a sum of two integer squares at all. A prime of the form $4\lambda - 1$ does not divide a sum of two relatively prime integer squares. Hence, if a prime divides a sum of two relatively prime integer squares, then it must have the form $4\lambda + 1$. (3)

The Main Result

We are now in a position to establish our main result. In what follows the symbol $x|y$ denotes x divides y, and $x \not| \, y$ denotes its negation. Also, BC, CA, AB or a, b, c denote the sides, as well as the lengths of the sides, of triangle ABC.

Theorem: If two sides of a primitive Heron triangle have a common divisor d, $d > 1$, then every prime factor p of d has the form $4\lambda + 1$.

Proof: Suppose, without loss of generality, that $\angle BAC$ has the greatest measure so that $a \geq b$, c, and the altitude AD lies within the primitive Heron triangle ABC. The area $\frac{1}{2}(BC)(AD)$ is an integer, so that AD is rational. As explained in the illustration of the previous section (see Figures 1 and 2) we arrive at the Pythagorean triangles $A'B'D'$ and $A'D'C'$ (enlarged by a scale factor ℓ if necessary). Hence, from (2),

$$\left.\begin{array}{lllll} A'B' & = & \ell c & = & k_1\left(m_1^2+n_1^2\right)\,, \\ B'D' & = & \ell BD & = & k_1(2m_1n_1) \;\text{ or }\; k_1\left(m_1^2-n_1^2\right)\,, \\ A'D' & = & \ell AD & = & k_1\left(m_1^2-n_1^2\right) \;\text{ or }\; k_1(2m_1n_1)\,, \\ A'C' & = & \ell b & = & k_2\left(m_2^2+n_2^2\right)\,, \\ D'C' & = & \ell DC & = & k_2(2m_2n_2) \;\text{ or }\; k_2\left(m_2^2-n_2^2\right)\,, \\ A'D' & = & \ell AD & = & k_2\left(m_2^2-n_2^2\right) \;\text{ or }\; k_2(2m_2n_2) \end{array}\right\} \tag{4}$$

A divisor d is composed of prime factors p, so that it is enough to establish the theorem for prime factors. Essentially there are two cases to consider.

Case I. $AB = c$, $AC = b$ have a common prime factor p. This implies

$$p \mid \ell c \;=\; k_1\left(m_1^2+n_1^2\right) \quad\text{and}\quad p \mid \ell b \;=\; k_2\left(m_2^2+n_2^2\right)\,. \tag{5}$$

If $p \mid \left(m_i^2+n_i^2\right)$ for $i = 1$ or 2, then, from (3), p has the form $4\lambda + 1$ and the assertion of the theorem follows. If not, then $p \mid k_1$ and $p \mid k_2$. In this case p must divide BC. To prove this, note that p is a common factor of b, c and that as a consequence of (5)

$$p^i \,|\, \ell \qquad\text{implies}\qquad p^{i+1} \,|\, k_1,\, k_2 \qquad i = 0,\,1,\,2,\,\ldots. \tag{6}$$

Now, according to (4), $B'C' = B'D' + D'C'$ constitutes a particular one of four conceivable combinations with coefficients k_1 and k_2 which depends on the formulas for the sides of the Pythagorean triangles $A'B'D'$ and $A'D'C'$. Consequently,

$$\ell BC \;=\; B'C' \;=\; k_1(\cdots) + k_2(\cdots)$$

implies $p \mid BC$ due to (6). This is not possible because ABC is primitive Heron.

Case II. $AB = c$, $BC = a$ have a common prime factor p.

Hence, $p \mid A'B'$, $B'C'$. Let us consider a specific representation of $B'C'$ to proceed with our argument. Thus, we have

$$p \,|\, k_1\left(m_1^2+n_1^2\right) \quad\text{and}\quad p \,|\, k_1(2m_1n_1) + k_2(2m_2n_2)\,.$$

If $p \mid (m_1^2 + n_1^2)$, then from (3) p has the form $4\lambda + 1$ and the assertion of the theorem follows. If $p \not| \ (m_1^2 + n_1^2)$, then $p \mid k_1$, and p may or may not have the form $4\lambda + 1$. However, $p \mid k_1$ and $p \mid k_1(2m_1n_1) + k_2(2m_2n_2)$ implies that $p \mid k_2(2m_2n_2)$. If $p \mid k_2$, then, as we have seen in Case I, $p \mid A'C' = k_2\,(m_2^2 + n_2^2)$, leading to $p \mid AC$, a ruled out situation. Hence, $p \mid 2m_2n_2$.

Now $p \neq 2$. This is because in a primitive Heron triangle precisely one side is even and the other two odd (you may wish to prove this fact as an exercise). Hence, $p \mid m_2$ or n_2 but not both because $\gcd(m_2, n_2) = 1$. Suppose $p \mid m_2$. Also, $p \mid k_1$ and $p \not| k_2$, n_2. Then

$$A'D' \;=\; k_1\,(m_1^2 - n_1^2) \;=\; k_2\,(m_2^2 - n_2^2)$$

is impossible because it has at the same time to contain and not to contain the factor p.

The other expressions for $B'C'$, such as $k_1(2m_1n_1) + k_2\,(m_2^2 - n_2^2)$ and Case III (in which the sides AC, BC may have a common prime factor), all lead to a similar discussion. Therefore, we omit them. This completes the proof of the Theorem.

Conclusion

In the course of our discussion we have also obtained the primitive Heron triangle $(45, 296, 325)$ in which two sides have the common factor 5. Have you noticed this? In fact we propose the following problem for the reader to solve:

Problem 1. Show that there is an infinite number of primitive Heron triangles in each of which two sides have the common factor 5.

The theorem has the following implication. In a primitive Heron triangle, if two sides have a composite common divisor, then it is composed of powers of primes precisely of the form $4\lambda+1$: $5, 13, 17, 25, \ldots$. One should observe that these integers are precisely the lengths of the hypotenuses of primitive Pythagorean triangles.

Here is yet another problem for the reader to solve:

Problem 2. Determine a primitive Heron triangle in which two sides have the common divisor 65.

Our discussion has shown that what is impossible in a primitive Pythagorean triangle is possible in some primitive Heron triangle. Also impossible in a primitive Pythagorean triangle is this: two sides cannot be squares. However, there are primitive Heron triangles in which two sides can be squares, $(289, 784, 975)$, $(841, 1369, 1122)$ for example. The reader is invited to resolve the following:

Open Problem.

Prove or disprove the existence of a primitive Heron triangle in which each side is a square.

The reader's attention is drawn to [7]. It shows the unsuspected dependence on Heron triangles of the solution of a problem in the rational plane, the exclusive Cartesian plane of points whose both coordinates are rational.

ACKNOWLEDGEMENT.

The author thanks the referee and the Editor for their comments and suggestions to improve the presentation.

References

[1] A.H. Beiler, *Recreations in the Theory of Numbers*, Dover Publications, New York, (1966), pp. 104-107, 49-53.

[2] D.M. Burton, *Elementary Number Theory*, Wm. C. Brown Publishers, Dubuque, Iowa (1989).

[3] L.E. Dickson, *History of the Theory of Numbers*, Vol. II, Chelsea, NY, (1971), pp. 191-201, 229-232.

[4] H. Flanders, A Tale of Two Squares — and Two Rings, *Math. Magazine* MAA, **58** (Jan. 1985), pp. 3-11.

[5] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford Univ. Press, (1971), pp. 297-316.

[6] W.J. Le Veque, *Elementary Theory of Numbers*, Dover Publications, NY, (1990), pp. 108-109.

[7] K.R.S. Sastry, Heron Problems, *Math. and Comput. Education*, **29** (spring 1995), pp. 192-202.

[8] K.R.S. Sastry, Heron Triangles: A New Perspective, *Aust. Math. Soc. Gazette*, **26** (Oct. 1999), pp. 160-168.

[9] K.R.S. Sastry, Heron Triangles: An Incenter Perspective, to appear in *Math. Magazine*, MAA.

K.R.S. Sastry
Jeevan Sandhya
Doddakalsandra Post
Raghuvanahalli
Bangalore 560 062, India

THE SKOLIAD CORNER

No. 51

R.E. Woodrow

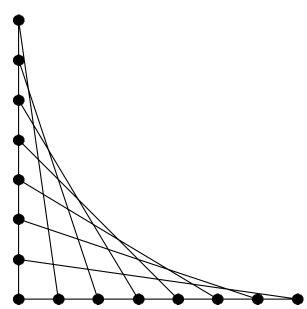
In the November 2000 number of the *Corner* we gave the problems of the Final Round of the British Columbia Colleges Junior High School Mathematics Contest, May 5, 2000. In this issue we give the solutions.

BRITISH COLUMBIA COLLEGES
Junior High School Mathematics Contest
Part A — Final Round — May 5, 2000

1. The last (ones) digit of a perfect square cannot be:

Solution. The answer is **E**. If we square the digits from 0 to 9 and consider the final digit of the square we get only the digits 0, 1, 4, 9, 6, and 5. Since there are no others, we see that 8 is NOT the final digit of any square.

2. Suppose a string art design is constructed by connecting nails on a vertical axis and on a horizontal axis by line segments as follows: The nail furthest from the origin on the vertical axis is connected to the nail nearest the origin on the horizontal axis. Then proceed toward the origin on the vertical axis and away on the horizontal axis as shown in the diagram. If this were done on a project with 10 nails on each axis, the number of points of intersection of line segments would be:



Solution. The answer is **A**. Each of the 10 straight lines intersects each of the others exactly once. This makes for 90 intersections; however, each of the intersections is counted twice in this approach, depending upon which of the two lines we consider first. To get the correct number of intersections we simply divide 90 by 2 to get 45.

3. Assume there is an unlimited supply of pennies, nickels, dimes, and quarters. An amount (in cents) which cannot be made using exactly 6 of these coins is:

Solution. The answer is **E**. Let us try successively to make up each of the given amounts using 6 coins:

$$\begin{aligned} 91 &= 1+5+10+25+25+25\,, \\ 87 &= 1+1+10+25+25+25\,, \\ 78 &= 1+1+1+25+25+25\,, \\ 51 &= 1+10+10+10+10+10\,. \end{aligned}$$

Thus each of the first 4 choices can be made up with 6 coins. In order to make up 49 we would need to use 4 pennies. This would require us to make up the total of 45 cents with only 2 coins, which is clearly impossible.

4. Given $x^2+y^2=28$ and $xy=14$, the value of x^2-y^2 equals:

Solution. The answer is **B**. First observe that

$$(x-y)^2 \;=\; x^2+y^2-2xy \;=\; 28-2(14) \;=\; 0\,.$$

This means that $x-y=0$; that is, $x=y$. In that event we clearly have $x^2-y^2=0$.

5. Given that $0<x<y<20$, the number of integer solutions (x,y) to the equation $23x+3y=50$ is:

Solution. The answer is **E**. Astute readers will notice that this problem and solution were presented in previous issues of the *Corner* as Question #10 of the British Columbia Colleges Senior High School Preliminary exam given late last year [2000 : 343, 476].

6. The numbers 1, 3, 6, 10, 15 ... are known as triangular numbers. Each triangular number can be expressed as $\frac{n(n+1)}{2}$ where n is a natural number. The largest triangular number less than 500 is:

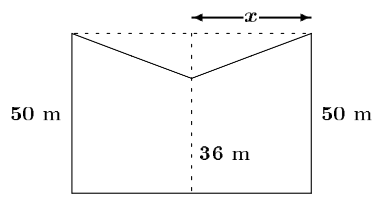
Solution. The answer is **C**. We are seeking the largest integer n such that

$$\frac{n(n+1)}{2} \;\le\; 500 \quad \text{or} \quad n(n+1) \;\le\; 1000$$

Since $32^2=1024$ we see that $n<32$. Checking $n=31$ we find $31\cdot 32=992$. Thus the integer n we seek is 31. The triangular number associated with this value of n is $\frac{1}{2}(992)=496$.

7. An 80 m rope is suspended at its two ends from the tops of two 50 m flagpoles. If the lowest point to which the mid-point of the rope can be pulled is 36 m from the ground, then the distance, in metres, between the flagpoles is:

Solution. The answer is **C**. In order for the rope to be at the lowest possible point, that point must be the middle of the rope. Thus we are faced with solving a right-angled triangle with hypotenuse 40 m and one side of $50 - 36 = 14$ m.



By the Theorem of Pythagoras the third side (x in the diagram at the right) is $\sqrt{40^2 - 14^2} = \sqrt{1404} = 6\sqrt{39}$. The distance between the two flagpoles is $2x = 12\sqrt{39}$.

8. At a certain party, the first time the door bell rang 1 guest arrived. On each succeeding ring two more guests arrived than on the previous ring. After 20 rings the number of guests at the party was:

Solution. The answer is **E**. Let a_n be the number of people who arrive at the n^{th} ring of the door bell. Then $a_n = 2n - 1$. Let b_n be the number of people who have arrived after the n^{th} ring of the door bell. Then we have

$$\begin{aligned} b_1 &= 1\,, \\ b_{n+1} &= b_n + a_{n+1} \quad \text{for } n \geq 1, \\ &= b_n + 2n + 1\,. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} b_1 &= 1\,, \\ b_{n+1} - b_n &= 2n + 1 \quad \text{for } n \geq 1. \end{aligned}$$

If we write out the first 20 of these we get

$$\begin{aligned} b_1 &= 1\,, \\ b_2 - b_1 &= 3\,, \\ b_3 - b_2 &= 5\,, \\ &\vdots \\ b_{20} - b_{19} &= 39\,. \end{aligned}$$

When we add all 20 of the above equations together we get

$$b_{20} = 1 + 3 + 5 + \cdots + 39 = \frac{1}{2}(20) \cdot (2 \cdot 1 + 19 \cdot 2) = 400\,,$$

where we have used the well-known formula for the sum of an arithmetic progression with n terms, having first term a and common difference d: $\frac{1}{2}n(2a + (n-1)d)$.

9. An operation $*$ is defined such that

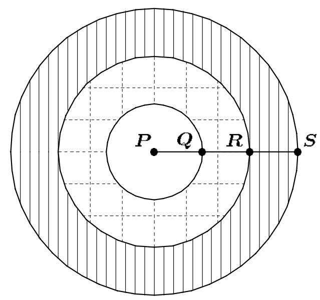
$$A * B \ = \ A^B - B^A$$

The value of $2 * (-1)$ is:

Solution. The answer is **C**. According to the definition of the operation $*$, we have

$$2 * (-1) \ = \ 2^{-1} - (-1)^2 \ = \ \frac{1}{2} - 1 = -\frac{1}{2}$$

10. Three circles with a common centre P are drawn as shown with $PQ = QR = RS$. The ratio of the area of the region between the inner and middle circles (shaded with squares) to the area of the region between the middle and outer circles (shaded with lines) is:



Solution. The answer is **D**. Let a be the length of PQ, QR, and RS. Then the radii of the 3 circles are a, $2a$, and $3a$. The area between the inner and middle circles is then $\pi(2a)^2 - \pi a^2 = 3\pi a^2$, and the area between the middle and outer circles is $\pi(3a)^2 - \pi(2a)^2 = 5\pi a^2$. Thus the ratio we want is $3\pi a^2 : 5\pi a^2 = \frac{3}{5}$.

Part B — Final Round — May 5, 2000

1. (a) How many 3-digit numbers can be formed using only the digits 1, 2, and 3 if both of the following conditions hold:

(i) repetition is allowed;

(ii) no digit can have a larger digit to its left.

(b) Repeat for a 4-digit number using the digits 1, 2, 3, and 4.

Solution. (a) For this part of the question, the simplest method is simply to list all the possible numbers. In increasing order they are:

$$111\ , 112\ , 113\ , 122\ , 123\ , 133\ , 222\ , 223\ , 233\ , \text{and } 333\ ,$$

for a total of 10 numbers.

(b) Again, most junior students will simply try to list all the possible integers. In increasing order they are:

$$1111\,,\,1112\,,\,1113\,,\,1114\,,\,1122\,,\,1123\,,\,1124\,,\,1133\,,\,1134\,,\,1144\,,$$

$$1222\,,\,1223\,,\,1224\,,\,1233\,,\,1234\,,\,1244\,,\,1333\,,\,1334\,,\,1344\,,\,1444\,,$$

$$2222\,,\,2223\,,\,2224\,,\,2233\,,\,2234\,,\,2244\,,\,2333\,,\,2334\,,\,2344\,,\,2444\,,$$

$$3333\,,\,3334\,,\,3344\,,\,3444\,, \quad \text{and} \quad 4444\,,$$

for a total of 35 numbers.

A more sophisticated approach (which can be generalized) follows: We first define $n(k,d)$ to be the number of k-digit integers ending with the digit d and satisfying the two conditions (i) and (ii) in the problem statement. Since a k-digit number ending with the digit d consists of appending the digit d to all $(k-1)$-digit numbers ending with a digit less than or equal to d, we have

$$n(k,d) \;=\; n(k-1,1) + n(k-1,2) + \cdots + n(k-1,d) \qquad (*)$$

Furthermore, we also have $n(1,d) = 1$ for all digits d and $n(k,1) = 1$ for all integers k. The relationship $(*)$ allows us to create the following table of values for $n(k,d)$:

k \ d	1	2	3	4
1	1	1	1	1
2	1	2	3	4
3	1	3	6	10
4	1	4	10	20

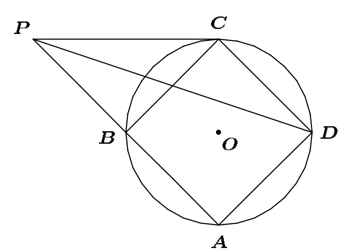
Each entry in the table is the sum of the entries in the previous row up to and including the column containing the given entry (note the presence of Pascal's Triangle in the table). From that table, the answers to parts (a) and (b) are:

(a) : $n(3,1) + n(3,2) + n(3,3) \;=\; 1+3+6 \;=\; 10\,,$

(b) : $n(4,1) + n(4,2) + n(4,3) + n(4,4) \;=\; 1+4+10+20 \;=\; 35\,.$

Clearly this table could have been extended to deal with any number k and with any digit $d \le 9$.

2. The square $ABCD$ is inscribed in a circle of radius one unit. ABP is a straight line, PC is tangent to the circle. Find the length of PD. Make sure you explain thoroughly how you got **all** the things you used to find your solution!

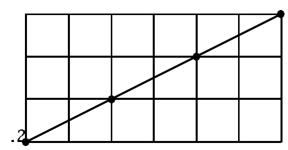


Solution. Since $ABCD$ is a square, the lines AC and BD are perpendicular. Since the circle had radius 1 unit, the Theorem of Pythagoras tells us that $AB = BC = CD = DA = \sqrt{2}$. The tangent PC at C is perpendicular to the diameter AC; thus $\angle PCB = 45°$. Since $PA \perp BC$ we also have $\angle CPB = 45°$. This makes $\triangle PBC$ isosceles, which means that $PB = BC = \sqrt{2}$. Applying the Theorem of Pythagoras to $\triangle APD$ we have

$$PD^2 \; = \; AP^2 + AD^2 \; = \; (2\sqrt{2})^2 + \sqrt{2}^2 \; = \; 8 + 2 \; = \; 10\,,$$

from which we see that $PD = \sqrt{10}$.

3. If a diagonal is drawn in a 3×6 rectangle, it passes through four vertices of smaller squares. How many vertices does the diagonal of a 45×30 rectangle pass through?



Solution. Since the 45×30 rectangle has its sides in the proportion $3 : 2$, we will consider first looking at a 3×2 rectangle, in which there are 2 vertices which lie on the diagonal. In the original 45×30 rectangle we need only consider the fifteen 3×2 rectangle which straddle the diagonal in question. The lower left of these has its lower leftmost vertex on the diagonal, and each of these 3×2 rectangles adds a further vertex to the count for its upper rightmost corner. This gives us a total of $1 + 15 = 16$ vertices on the diagonal.

4. Let a and b be any real numbers. Then $(a - b)$ is also a real number, and consequently $(a - b)^2 \geq 0$. Expanding gives $a^2 - 2ab + b^2 \geq 0$. If we add $2ab$ to both sides of the inequality, we get $a^2 + b^2 \geq 2ab$. Thus, for any real numbers a and b, we have $a^2 + b^2 \geq 2ab$.

Prove that for any real numbers a, b, c, d:

(a) $2abcd \;\leq\; b^2c^2 + a^2d^2$.

(b) $6abcd \;\leq\; a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2$.

Solution. (a) We will use the proof in the problem statement as a model. Consider $bc - ad$. Clearly $(bc - ad)^2 \geq 0$. Expanding gives

$$b^2c^2 - 2abcd + a^2d^2 \;\geq\; 0\,.$$

This is easily rearranged to yield $b^2c^2 + a^2d^2 \geq 2abcd$.

(b) We will use part (a) to prove part (b). Since a, b, c, and d are arbitrary real numbers, the inequality in part (a) remains true for any rearrangement of the letters; in particular we have:

$$\begin{aligned} 2abcd &\leq b^2c^2 + a^2d^2\,, \\ 2abdc &\leq b^2d^2 + a^2c^2\,, \\ 2adcb &\leq d^2c^2 + a^2b^2\,. \end{aligned}$$

Recognizing that multiplication is commutative for real numbers, we can reorganize the products in each of the above inequalities and sum the three inequalities to get the desired result.

5. A circular coin is placed on a table.Then identical coins are placed around it so that each coin touches the first coin and its other two neighbours.

(a) If the outer coins have the same radius as the inner coin, show that there will be exactly 6 coins around the outside.

(b) If the radius of all 7 coins is 1, find the total area of the spaces between the inner coin and the 6 outer coins.

Solution. (a) Let us place 2 (outer) coins next to the original coin so that they touch each other. Then the centres of the 3 coins form an equilateral triangle with side length equal to twice the radius of a single coin. Therefore the angle between the centres of the 2 (outer) coins measured at the centre of the first coin is 60°. Since 6 such angles make up a full revolution around the inner coin, we can have exactly 6 outer coins each touching the original (inner) coin and also touching its other two neighbours.

(b) There are 6 non-overlapping spaces whose areas we must add; each is found between 3 coins which simultaneously touch other, and whose centres form the equilateral triangle mentioned in part (a) above. This equilateral triangle has side length 2, since we are given the radii of the coins as 1. Our strategy to compute the area of one such space is to find the area of the equilateral triangle and subtract the areas of the 3 circular sectors found within the triangle. The altitude of the equilateral triangle with side length 2 can be easily found (Theorem of Pythagoras) as $\sqrt{3}$. Thus the area of the triangle itself is $\frac{1}{2} \cdot 2 \cdot \sqrt{3} = \sqrt{3}$. The area of a single coin is $\pi \cdot 1^2 = \pi$. The circular sectors within the equilateral triangle are each one-sixth of the area of the coin; there are 3 such sectors which gives us a total area of one-half the area of a single coin to be subtracted from the area of the equilateral triangle. Thus the area of a single space is $\sqrt{3} - (\pi/2)$. Since there are 6 such spaces, we have a total area of $6\sqrt{3} - 3\pi$ square units.

That completes the *Skoliad Corner* for this issue. Send me suitable contest materials for future use in ***CRUX with MAYHEM***.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to Mathematical Mayhem, Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada. M65 3G3. The electronic address is

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The rest of the staff consists of Adrian Chan (Harvard University), Jimmy Chui (University of Toronto), Donny Cheung (University of Waterloo), and David Savitt (Harvard University)

From the Editor-in-Chief

First, the Board of ***Crux Mathematicorum with Mathematical Mayhem*** extends a warm welcome to the new MAYHEM Editor, Shawn Godin, and to the new MAYHEM Assistant Editor, Chris Cappadocia. Shawn has been a member of the ***CRUX with MAYHEM*** family for many years. You can read his first editorial below.

Our sincere thanks go to the retiring editors, Naoki Sato and Cyrus Hsia. Both have moved on to other things — Naoki is an Actuarial Student and Cyrus is a Medical Student. We wish them every success in their new careers. They were instrumental in the smooth transition from two independent journals to the combined journal, and, as Editor-in-Chief, I thank them most sincerely for all their efforts and achievements.

Bruce Shawyer

Editorial

It is the beginning of a new volume and the start of my term as editor of Mayhem. As a brief introduction, I am a high school math and physics teacher in Orleans, east of Ottawa. I have been a reader of Crux for a number of years and am looking forward to working with all of the staff of Crux Mathematicorum with Mathematical Mayhem.

At this point I want to thank our outgoing staff. Naoki Sato has been on the staff of Mathematical Mayhem for a number of years. As editor, he has seen the marriage of Mayhem with Crux and has kept the original vision of Mayhem intact. He has been a great help, so far, to my transition to the editor's chair (and I hope he will be there to answer those frantic emails for a while yet!). Cyrus Hsia is also stepping down from his post as Mayhem's assistant editor. Together, these two guys have kept Mayhem the quality journal that it is. They will be greatly missed.

With Cyrus stepping down we have a new assistant editor, Chris Cappadocia. Chris is a former student of mine, from when I taught in North Bay, who is now in his first year of studies at the University of Waterloo. Chris is a very creative person, passionate about mathematics. He will be an asset to Mayhem, and I am looking forward to working with him.

A couple of months ago, some of the staff of Mayhem, old and new, sat down at the Fields Institute to discuss the direction of Mayhem. Our main conclusion was that Mayhem was to keep its focus as a journal for **students**. Since Mayhem now resides within Crux, we have decided to trim some of the areas where Crux and Mayhem overlap, to allow space for other material that will be of interest to the readers of Mayhem. Keep your eyes open for editorials in upcoming issues describing some of these new features.

Over the next year we will be phasing in some changes to Mayhem. Let us know what you think of them, and tell us about anything **you** would like to see. Remember, Mayhem is **your** journal. Help us to make it as good as it can be.

Shawn Godin

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 2 of 2002.

High School Problems

Editor: Adrian Chan, 1195 Harvard Yard Mail Center, Cambridge, MA, USA 02138-7501 <ahchan@fas.harvard.edu>

H281 *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Suppose the monic polynomial $A(z) = \sum_{k=0}^{n} a_k z^k$ can be factored into $(z - z_1)(z - z_2)\cdots(z - z_n)$, where $z_1, z_2, \cdots, z_n$ are positive real numbers. Prove that $a_1 a_{n-1} \geq n^2 a_0$.

H282. Let $ABCD$ be a cyclic quadrilateral such that its diagonals are perpendicular. Let E be the intersection of AC and BD. It is known that $AE + ED = BE + EC$. Show that $ABCD$ is a trapezoid.

H283. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a_1, a_2, ..., a_n be positive real numbers in arithmetic progression. Prove that

$$\sum_{k=1}^{n} \frac{1}{a_k a_{n-k+1}} > \frac{4n}{(a_1 + a_n)^2}.$$

H284. Prove that for any positive integer n,

$$1 \geq \frac{n^n}{(n!)^2} \geq \frac{(4n)^n}{(n+1)^{2n}}.$$

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A257. Given a quadrilateral $ABCD$, show that

$$|AB| \cdot |CD| + |BC| \cdot |AD| \geq |AC| \cdot |BD|.$$

When does equality hold?

A258. Is it possible to partition all positive integers into disjoint sets A and B such that

(i) no three numbers of A form an arithmetic progression, and

(ii) no infinite non-constant arithmetic progression can be formed by numbers of B?

(1996 Baltic Way)

A259. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(n^2) = f(n+m)f(n-m) + m^2$$

for all $m, n \in \mathbb{Z}$.

A260.

Characterize the set of Pythagorean triples (integers (a, b, c) such that $a^2 + b^2 = c^2$) which do not contain a multiple of 5.

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C99. Find all collections of polynomials p_{11}, p_{12}, p_{21}, p_{22} with complex coefficients satisfying the relation

$$\begin{pmatrix} p_{11}(XY) & p_{12}(XY) \\ p_{21}(XY) & p_{22}(XY) \end{pmatrix} = \begin{pmatrix} p_{11}(X) & p_{12}(X) \\ p_{21}(X) & p_{22}(X) \end{pmatrix} \cdot \begin{pmatrix} p_{11}(Y) & p_{12}(Y) \\ p_{21}(Y) & p_{22}(Y) \end{pmatrix} .$$

C100. *Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let x_1, x_2, $\ldots$, x_n be positive real numbers, let $S = \sum_{k=1}^{n} x_k$, and suppose that $(n-1)x_k < S$ for all k. Prove that

$$\prod_{j=1}^{n} (S - (n-1)x_k) \;\leq\; \prod_{j=1}^{n} x_j \,.$$

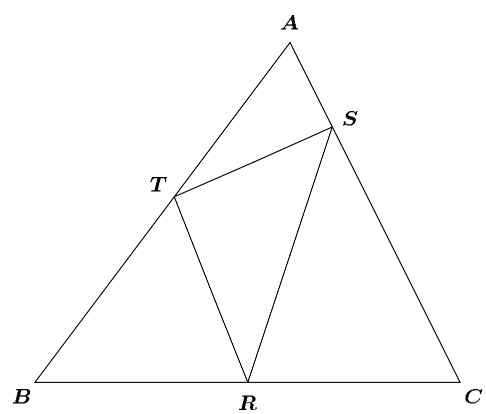
When does equality occur?

Problem of the Month

Jimmy Chui, student, University of Toronto

Problem. In triangle ABC, the points R, S, and T lie on the line segments BC, CA, and AB, respectively, such that R is the mid-point of BC, $CS = 3SA$, and $\frac{AT}{TB} = \frac{p}{q}$. If w is the area of $\triangle CRS$, x is the area of $\triangle RBT$, z is the area of $\triangle ATS$, and $x^2 = wz$, then what is the value of $\frac{p}{q}$?

(1997 Fermat, Problem 25)



Solution. The area of a triangle is one half of *base* times *height*. Let A be the area of $\triangle ABC$ and let $r = \frac{p}{q}$.

The ratio of the bases of $\triangle RBT$ and $\triangle ABC$ is $\frac{BR}{BC} = \frac{1}{2}$, and the ratio of the heights is $\frac{BT}{BA} = \frac{q}{p+q} = \frac{q/q}{p/q+q/q} = \frac{1}{r+1}$. Thus, the area of $\triangle RBT$ is $x = \frac{1}{2} \cdot \frac{1}{r+1} A$.

The ratio of the bases of $\triangle CRS$ and $\triangle ABC$ is $\frac{RC}{BC} = \frac{1}{2}$, and the ratio of the heights is $\frac{CS}{CA} = \frac{3}{4}$. Thus, the area of $\triangle CRS$ is $w = \frac{1}{2} \cdot \frac{3}{4} A = \frac{3}{8} A$.

Now reorient the triangle so that CA is the base of ABC. The ratio of the bases of $\triangle ATS$ and $\triangle ABC$ is $\frac{AS}{AC} = \frac{1}{4}$. The ratio of the heights is $\frac{AT}{AB} = \frac{p}{p+q} = \frac{p/q}{p/q+q/q} = \frac{r}{r+1}$. Thus, the area of $\triangle ATS$ is $z = \frac{1}{4} \cdot \frac{r}{r+1} A$.

Therefore, $x^2 = \frac{1}{4(r+1)^2} A^2$ and $wz = \frac{3}{8} \cdot \frac{r}{4(r+1)} A^2$. We must then have $\frac{1}{4(r+1)^2} = \frac{3}{8} \cdot \frac{r}{4(r+1)}$, or, after simplifying, $3r^2 + 3r - 8 = 0$. Solving this quadratic equation leads to the two values $r = \frac{-3 \pm \sqrt{105}}{6}$. Choosing the negative sign leads to a negative value of r, which makes no sense. Thus, we must have $r = \frac{-3+\sqrt{105}}{6}$. This is our value for $\frac{p}{q}$.

Astonishing Pairs of Numbers

Richard Hoshino, student, University of Waterloo

During one dull psychology class, I mindlessly scribbled in my notes: $1+2+3+4+5 = 15$. It was nothing overly profound, but I thought it was rather interesting that the sum of the numbers from 1 to 5, inclusive, resulted in the digit 1 followed by the digit 5. Curious to see what other combinations exhibited similar properties, I fooled around with some numbers and soon found out that $2+\ldots+7 = 27$, and $4+\ldots+29 = 429$. Thus, this motivated the following problem.

We say that an ordered pair of positive integers (a, b) with $a < b$ is ***astonishing*** *if the sum of the integers from a to b, inclusive, is equal to the digits of a followed by the digits of b. Determine all astonishing ordered pairs.*

Surprisingly, there are many beautiful patterns that arise from this problem, and we shall describe them in this article. I was curious to see if there was an ordered pair (a, b) where b has exactly 1999 digits, and while attempting to solve this problem, I discovered some extremely neat results. Hopefully this article will illustrate that often the simplest of ideas can lead to surprising and extraordinary results. In my case, however, this was completely by accident.

Our problem can be reformulated as follows: find all solutions in positive integers a and b, with $a + (a+1) + \ldots + (b-1) + b = a \cdot 10^n + b$, where b is an integer with exactly n digits.

So our equation becomes:

$$\begin{aligned} \frac{b(b+1)}{2} - \frac{(a-1)a}{2} &= a \cdot 10^n + b\,, \\ b(b+1) - a(a-1) &= 2a \cdot 10^n + 2b\,, \\ b^2 + b - a^2 + a &= 2a \cdot 10^n + 2b\,, \\ b^2 - b - (a^2 + (2 \cdot 10^n - 1)a) &= 0\,. \end{aligned}$$

And by the quadratic formula, we find that

$$\begin{aligned} b &= \frac{1 \pm \sqrt{1 + 4(a^2 + (2 \cdot 10^n - 1)a)}}{2} \\ &= \frac{1 \pm \sqrt{(2a + (2 \cdot 10^n - 1))^2 - ((2 \cdot 10^n - 1)^2 - 1)}}{2}\,. \end{aligned}$$

We want b to be an integer, so we require that the discriminant be a perfect square, specifically an odd perfect square. Thus, let

$$(2a + (2 \cdot 10^n - 1))^2 - ((2 \cdot 10^n - 1)^2 - 1) \;=\; m^2\,,$$

where m is a positive integer. Then, $b = \frac{1+m}{2}$. Note that we can get rid of the $\pm$ sign because we require $b > 0$. Thus we have:

$$\begin{aligned}
(2a + (2 \cdot 10^n - 1))^2 - ((2 \cdot 10^n - 1)^2 - 1) &= m^2\,,\\
(2a + (2 \cdot 10^n - 1))^2 - m^2 &= (2 \cdot 10^n - 1)^2 - 1\,,\\
(2a + (2 \cdot 10^n - 1) + m)(2a + (2 \cdot 10^n - 1) - m) &\\
&= ((2 \cdot 10^n - 1) + 1)((2 \cdot 10^n - 1) - 1)\,,\\
(2a + 2 \cdot 10^n + m - 1)(2a + 2 \cdot 10^n - m - 1) &\\
&= 2 \cdot 10^n \cdot (2 \cdot 10^n - 2)\,,\\
(2a + 2 \cdot 10^n + m - 1)(2a + 2 \cdot 10^n - m - 1) &\\
&= 2^{n+2} \cdot 5^n \cdot (10^n - 1)\,.
\end{aligned}$$

Let $x = 2a + 2 \cdot 10^n + m - 1$ and $y = 2a + 2 \cdot 10^n - m - 1$. Then we have $xy = 2^{n+2} \cdot 5^n(10^n - 1)$ and $x + y = 4a + 4 \cdot 10^n - 2$. But then $x + y$ is a multiple of 2, but not 4, so both x and y cannot be multiples of 4. Furthermore, $x + y$ is even, so we cannot have x being odd and y being even, or vice-versa. It follows that the only possibilities are to have $(x, y) = (2p, 2^{n+1}q)$ or $(x, y) = (2^{n+1}q, 2p)$, where $pq = 5^n(10^n - 1)$.

Hence, $4a + 4 \cdot 10^n - 2 = x + y = 2^{n+1}q + 2p$, and so

$$a \;=\; \frac{2^{n+1}q + 2p + 2 - 4 \cdot 10^n}{4} \;=\; 2^{n-1}q - 10^n + \frac{p+1}{2}\,.$$

Now, $b = \frac{1+m}{2}$ and $2m = x - y$, so that $b = \frac{2+x-y}{4}$. But b is a positive integer. Thus, we require $x - y$ to be positive; that is, $x - y = |2^{n+1}q - 2p|$. Note that the formula is correct for both of our possibilities for (x, y) detailed above. Thus, we have:

$$b \;=\; \frac{1 + |2^n q - p|}{2}\,.$$

Hence, for each n, we need to cycle through all possible ordered pairs (p, q) so that $pq = 5^n(10^n - 1)$, and see if this ordered pair (p, q) will give a solution (a, b), where $0 < a < b$ and b has exactly n digits.

For example, for $n = 1$, we have $pq = 45$, so that the only possibilities for (p, q) are $(1, 45)$, $(3, 15)$, $(5, 9)$, $(9, 5)$, $(15, 3)$, $(45, 1)$. Substituting into our formula for a and b, we find that the corresponding pairs (a, b) are $(36, 45)$, $(7, 14)$, $(2, 7)$, $(0, 1)$, $(1, 5)$, $(14, 22)$, respectively. Only $(1, 5)$ and $(2, 7)$ satisfy our requirements, so these are the only astonishing pairs with $n = 1$.

For any $n > 1$, the number $pq = 5^n(10^n - 1)$ has many factors, so that it is quite time consuming to look for each possible pair (p, q). However, with the power of a computer algebra system, such as MAPLE, the calculations are very easy.

With the aid of MAPLE, we list all astonishing pairs (a, b), where $n \leq 5$.

$$\begin{array}{lcll} n & = & 1 & : & (1,5),\ (2,7) \\ n & = & 2 & : & (4,29),\ (13,53),\ (18,63),\ (33,88),\ (35,91) \\ n & = & 3 & : & (7,119),\ (78,403),\ (133,533),\ (178,623) \\ n & = & 4 & : & (228,2148),\ (273,2353),\ (388,2813),\ (710,3835), \\ & & & & (1333,5333),\ (1701,6076),\ (1778,6223), \\ & & & & (2737,7889),\ (3273,8728),\ (3563,9163) \\ n & = & 5 & : & (3087,25039),\ (3478,26603),\ (12488,51513), \\ & & & & (13333,53333),\ (14208,55168),\ (17778,62223), \\ & & & & (31463,85338),\ (36993,93633). \end{array}$$

Look carefully at these numbers, we have some very interesting patterns here. The pairs $(1, 5)$, $(13, 53)$, $(133, 533)$, $(1333, 5333)$, $(13333, 53333)$ are all astonishing. It is very likely that this pattern continues indefinitely and, as we shall see, this is indeed the case. We shall also find some other incredible sequences of astonishing pairs, and discover some remarkable properties of such sequences.

Let us attempt to find formulas that generate astonishing pairs. Assume that $q = 5^n r$. Since $pq = 5^n(10^n - 1)$, we have $p = \frac{10^n-1}{r}$. Then we can solve for a and b in terms of r.

$$\begin{aligned} a &= 2^{n-1}q - 10^n + \frac{p+1}{2} \\ &= 2^{n-1} \cdot 5^n r - 10^n + \frac{\frac{10^n-1}{r} + 1}{2} \\ &= 10^n \cdot \frac{r}{2} - 10^n + \frac{10^n + r - 1}{2r} \\ &= \frac{10^n}{2}\left(r + \frac{1}{r} - 2\right) + \frac{r-1}{2r}, \\ b &= \frac{1 + |2^n q - p|}{2} \\ &= \frac{1 + |2^n \cdot 5^n r - \frac{10^n-1}{r}|}{2} \\ &= \frac{1 + |10^n r - \frac{10^n-1}{r}|}{2} \\ &= \frac{1 + |10^n(r - \frac{1}{r}) + \frac{1}{r}|}{2}. \end{aligned}$$

Now we shall find values of r such that a and b are integers for all n. In order for (a, b) to be an astonishing ordered pair, we require b to have exactly n digits; that is, we require $10^{n-1} \leq b < 10^n$. If we can find an r such that this inequality holds for all n, then we will generate an infinite set of astonishing ordered pairs (a, b), since each integer n will give us one astonishing ordered pair.

For large n, $b = \frac{1+\left|10^n\left(r-\frac{1}{r}\right)+\frac{1}{r}\right|}{2}$ can be approximated as $\left|\frac{10^n}{2}\left(r-\frac{1}{r}\right)\right|$, since 1 and $\frac{1}{r}$ are extremely small quantities, compared with $10^n\left(r-\frac{1}{r}\right)$. Simplifying $10^{n-1} \leq \left|\frac{10^n}{2}\left(r-\frac{1}{r}\right)\right| < 10^n$, we get $\frac{1}{5} \leq \left|r-\frac{1}{r}\right| < 2$.

Solving this inequality, we find that we require $\sqrt{2}-1 < r \leq \frac{\sqrt{101}-1}{10}$ or $\frac{\sqrt{101}+1}{10} \leq r < \sqrt{2}+1$. (Note: we require $r > 0$, since otherwise p and q will be negative).

Since p and q must be (odd) integers, r must be a rational number of the form $\frac{x}{y}$, where y divides 5^n and x divides $10^n - 1$. If for a certain n, y does not divide 5^n or x does not divide $10^n - 1$, then we will not get an astonishing pair for that value of n.

For example, we can have $r = \frac{3}{5}$, since 3 divides $10^n - 1$ for all n, 5 divides 5^n for all n, and $\sqrt{2}-1 < \frac{3}{5} \leq \frac{\sqrt{101}-1}{10}$.

Substituting $\frac{3}{5}$ into our formula for a and b, we find, upon simplification, that:

$$\begin{aligned} a &= \frac{2 \cdot 10^n - 5}{15}, \\ b &= \frac{8 \cdot 10^n - 5}{15}. \end{aligned}$$

By our work above, we know that each ordered pair (a, b) generated by the formula above is astonishing. Substituting in $n = 1, 2, 3, \ldots,$ we generate an infinite number of astonishing ordered pairs:

$$(1, 5),\ (13, 53),\ (133, 533),\ (1333, 5333),\ (13333, 53333), \ldots.$$

So, as hypothesized earlier, this pattern does indeed continue indefinitely. Analyzing this problem in this manner, we can generate other sequences of astonishing ordered pairs.

Let $r = \frac{9}{5}$. This number r ensures that p and q are integers for $n \geq 2$, and $\frac{\sqrt{101}+1}{10} \leq \frac{9}{5} < \sqrt{2}+1$. Hence for every $n \geq 2$, we can find an astonishing ordered pair (a, b) where $r = \frac{9}{5}$. Substituting this value of r into our formula for a and b, we find that:

$$\begin{aligned} a &= \frac{2(4 \cdot 10^n + 5)}{45}, \\ b &= \frac{7(4 \cdot 10^n + 5)}{45}. \end{aligned}$$

For each $n \geq 2$, we get an astonishing ordered pair:

$$(18, 63), (178, 623), (1778, 6223), (17778, 62223), (177778, 622223), \ldots$$

As you notice, there is a really interesting pattern in this sequence as well. Notice that to generate astonishing pairs, we repeatedly add a 7 to each a and a 2 to each b. Is it not weird then that $\frac{b}{a} = \frac{7}{2}$? In addition, $(2, 7)$ is an astonishing pair. I wonder if that is a coincidence?

Let us try some other values of r that satisfy the requirements that we specified. Let us try $r = \frac{99}{125}$. Notice that 125 divides 5^n for each $n \geq 3$, and 99 divides $10^n - 1$ for each even n. In addition, $\sqrt{2} - 1 < \frac{99}{125} \leq \frac{\sqrt{101}-1}{10}$. Thus, our formula for a and b will only give us astonishing pairs for $n = 4$, 6, 8, 10, $\ldots$.

Our formula for $r = \frac{99}{125}$ is:

$$\begin{aligned} a &= \frac{13(26 \cdot 10^n - 125)}{12375}, \\ b &= \frac{13(224 \cdot 10^n - 125)}{12375}. \end{aligned}$$

For $n = 4, 6, 8, 10, \ldots$, we find that the corresponding astonishing pairs are:

$$(273, 2353), (27313, 235313), (2731313, 23531313), \\ (273131313, 2353131313), \ldots.$$

In this sequence, we add 13 at the end of each a and b to generate the next astonishing pair. Is it not weird that the number 13 appears in the formula for both a and b? Is that just a coincidence, or is there a reason why this is true?

Let us try one more value of r. Let us try $r = \frac{999}{625}$. We have $\frac{\sqrt{101}+1}{10} \leq \frac{999}{625} < \sqrt{2} + 1$. Since 625 divides 5^n for each $n \geq 4$, and 999 divides $10^n - 1$ if and only if n is a multiple of 3, our formula for a and b will only give us astonishing pairs for $n = 6, 9, 12, 15, \ldots$.

Our formula for $r = \frac{999}{625}$ is:

$$
\begin{aligned}
a &= \frac{187(374 \cdot 10^n + 625)}{624375}, \\
b &= \frac{812(374 \cdot 10^n + 625)}{624375}.
\end{aligned}
$$

For $n = 6, 9, 12, 15, \ldots$, we find that the corresponding astonishing pairs are:

$$
\begin{gathered}
(112013, 486388),\ (112012813, 486387188), \\
(112012812813, 486387187188), \\
(112012812812813, 486387187187188), \\
(112012812812812813, 486387187187187188), \ldots .
\end{gathered}
$$

In this sequence, we add the number 281 near the end of each a to get the next term and we add the number 718 near the end of each b to get the next term. Notice that $\frac{b}{a} = \frac{812}{187}$. Now is it just a remarkable coincidence, or is there a reason why 281 and 718 are cyclic permutations of the number 812 and 187? Also, would it not be simply incredible if $(281, 718)$ is an astonishing ordered pair? Unfortunately if we add up the numbers from 281 to 718 inclusive we get the number 218781, which is not astonishing, but remarkably, it is a permutation of the number 281718; that is, it is almost astonishing! Now is *this* a coincidence, or is there some deep mathematical explanation as to why this is the case?

Try some other rational values of r, and generate some more patterns of astonishing sequences. For example, what sequences do $r = \frac{111}{125}$ and $r = \frac{9999}{15625}$ generate? Are there any interesting mathematical observations that you can make from looking at the patterns?

I have left many questions unanswered in this article, as the mathematics involved in answering these questions almost definitely extends beyond what I currently know. Possibly you will be able to extend the ideas in this article further. Is it not quite strange that we were able to derive all these interesting results, and it was motivated by the simple observation that $1 + 2 + 3 + 4 + 5 = 15$. That is quite *astonishing*, is it not?

Richard Hoshino
Faculty of Mathematics
University of Waterloo
Waterloo, Ontario
rhoshino@student.math.uwaterloo.ca

A Classical Inequality

Vedula N. Murty

Problem.
Let A, B, and C denote the angles of a triangle ABC. The following inequality is well known:

$$1 \ < \ \cos A + \cos B + \cos C \ \leq \ \frac{3}{2} \,. \qquad (1)$$

We present two solutions to the left-hand side of the inequality and three solutions to the right-hand side inequality and draw the attention of the students to a wrong proof usually given to prove the right-hand side inequality. One of the proofs presented for the left-hand side inequality is believed to be new.

Proof 1.
Consider the identity given below which is easily verified.

$$\sum a(b^2 + c^2 - a^2) \ \equiv \ (a+b-c)(b+c-a)(c+a-b) + 2abc \,. \qquad (2)$$

Here, a, b, and c denote the side lengths of a triangle, and the sum is cyclical over a, b, and c. Dividing both sides of equation (2) by $2abc$ we obtain

$$\sum \frac{b^2 + c^2 - a^2}{2bc} \ = \ \frac{(a+b-c)(b+c-a)(c+a-b)}{2abc} + 1 \,. \qquad (3)$$

Noting that the left-hand side of (3) is $\cos A + \cos B + \cos C$, we immediately see that $\cos A + \cos B + \cos C > 1$, since the right-hand side of (3) is greater than 1. We believe this proof is new.

Let $x = a + b - c$, $y = b + c - a$, and $z = c + a - b$. Then x, y, and z are all positive. This implies that $a = (z + x)/2$, $b = (x + y)/2$, and $c = (y + z)/2$. The Arithmetic-Geometric Inequality gives us

$$(x+y)(y+z)(z+x) \ \geq \ 8xyz \,. \qquad (4)$$

This implies that

$$\cos A + \cos B + \cos C \ \leq \ \frac{3}{2} \,.$$

This completes the proof of the classical inequality (1).

Proof 2.

We now present a trigonometric proof of the classical inequality. The following are well known trigonometric identities given in standard text books on trigonometry. The reference given at the end of this paper is an excellent text book on trigonometry.

$$\cos A + \cos B + \cos C = 1 + 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right), \quad (5)$$

$$r = 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right), \quad (6)$$

$$OI^2 = R^2 - 2Rr. \quad (7)$$

Notice that (7) implies that $R^2 - 2Rr \geq 0$, which implies that

$$0 < \frac{r}{R} \leq \frac{1}{2}. \quad (8)$$

Equations (5) and (6) imply that

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}. \quad (9)$$

Equations (8) and (9) prove the classical inequality. The notation used above is standard and is familiar to readers of ***CRUX with MAYHEM***.

Proof 3.

We now use Jensen's inequality, which states that

$$w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) \leq f(w_1x_1 + w_2x_2 + w_3x_3), \quad (10)$$

where the w_i are positive with a sum of 1, and $f(x)$ is concave down throughout its domain.

Quite a few students use (10) with $w_1 = w_2 = w_3 = 1/3$ and $f(x) = \cos x$. Unfortunately, $\cos x$ is not concave down throughout $(0, \pi)$.

To circumvent this difficulty we first prove the inequality

$$\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right) \leq \frac{3}{2}. \quad (11)$$

To see this, simply take $w_1 = w_2 = w_3 = 1/3$ and $f(x) = \sin(x/2)$ and use (10) with $x_1 = A$, $x_2 = B$, and $x_3 = C$. We immediately obtain (11).

We now establish the following inequality:

$$\cos A + \cos B + \cos C \leq \sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right). \quad (12)$$

First,

$$\begin{aligned}\cos B + \cos C &= 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) \\ &= 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) \le 2\sin\frac{A}{2}\,. \qquad (13)\end{aligned}$$

Similarly,

$$\cos C + \cos A \le 2\sin\frac{B}{2}\,, \qquad (14)$$

$$\cos A + \cos B \le 2\sin\frac{C}{2}\,. \qquad (15)$$

Adding (13), (15), and (15) we obtain (12). This completes the proof of the classical inequality.

Reference

S. L. Loney, "Plane Trigonometry Part 1", Metric Edition, Radha Publishing House, Calcutta.

Vedula N. Murty
Penn State University
Harrisburg
PA, USA

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"$\times$11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 September 2001**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2601. *Proposed by Michel Bataille, Rouen, France.*

Sequences $\{u_n\}$ and $\{v_n\}$ are defined by $u_0 = 4$, $u_1 = 2$, and for all integers $n \geq 0$, $u_{n+2} = 8t^2 u_{n+1} + \left(t - \frac{1}{2}\right) u_n$, $v_n = u_{n+1} - u_n$. For which t is $\{v_n\}$ a non-constant geometric sequence?

2602*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For integers a, b and c, let $Q(a,b,c)$ be the set of all numbers $an^2 + bn + c$, where $n \in \mathbb{N} = \{0, 1, \ldots, \}$.

(a) Show that $Q(6, 3, -2)$ is square-free.

(b) Determine other infinite sets $Q(a,b,c)$ with the same property.

2603. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*

Suppose that A, B and C are the angles of a triangle. Prove that

$$\sin A + \sin B + \sin C \;\leq\; \sqrt{\frac{15}{4} + \cos(A-B) + \cos(B-C) + \cos(C-A)}\,.$$

2604. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) Determine the upper and lower bounds of $\dfrac{a}{a+b}+\dfrac{b}{b+c}-\dfrac{a}{a+c}$ for all positive real numbers a, b and c.

(b)* Determine the upper and lower bounds (as functions of n) of

$$\sum_{j=1}^{n-1}\frac{x_j}{x_j+x_{j+1}}-\frac{x_1}{x_1+x_n}$$

for all positive real numbers x_1, x_2, $\ldots$, x_n.

2605. *Proposed by K.R.S. Sastry, Bangalore, India.*

In triangle ABC, with median AD and internal angle bisector BE, we are given $AB=7$, $BC=18$ and $EA=ED$. Find AC.

2606. *Proposed by K.R.S. Sastry, Bangalore, India.*

A Gergonne cevian connects the vertex of a triangle to the point at which the incircle is tangent to the opposite side.

Determine the unique triangle ABC (up to similarity) in which the Gergonne cevian BE bisects the median AM, and the Gergonne cevian CF bisects the median NB.

2607. *Proposed by Václav Konečný, Ferris State University, Big Rapids, MI, USA.*

(a) Suppose that $q>p$ are odd primes such that $q=pn+1$, where n is an integer greater than 1. Let z be a complex number such that $z^q=1$.

Prove that $\dfrac{z^p-1}{z^p+1} \;=\; \displaystyle\sum_{j=1}^{q-1}(-1)^{\lfloor\frac{j-1}{p}\rfloor}z^j$.

(b) Suppose that $q>3$ is an odd prime such that $q=3n+2$, where n is an integer greater than 1. Let z be a complex number such that $z^q=1$.

Prove that $\dfrac{z^3-1}{z^3+1} \;=\; \displaystyle\sum_{j=1}^{q-1}(-1)^{\lfloor\frac{j-3}{3}\rfloor}z^j$.

2608*. *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that x, y, $z\ge 0$ and $x^2+y^2+z^2=1$. Prove or disprove that

(a) $1 \;\le\; \dfrac{x}{1-yz}+\dfrac{y}{1-zx}+\dfrac{z}{1-xy} \;\le\; \dfrac{3\sqrt{3}}{2}$;

(b) $1 \;\le\; \dfrac{x}{1+yz}+\dfrac{y}{1+zx}+\dfrac{z}{1+xy} \;\le\; \sqrt{2}$.

2609. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

A convex polygon P_n ($n \geq 4$) has the following property:

> the $n-3$ diagonals emanating from each of the n vertices of P_n divide the corresponding angle of P_n into $n-2$ equal parts.

Determine the shape of P_n.

2610. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*

Let $\{F_n\}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Prove that, for $n \geq 1$,

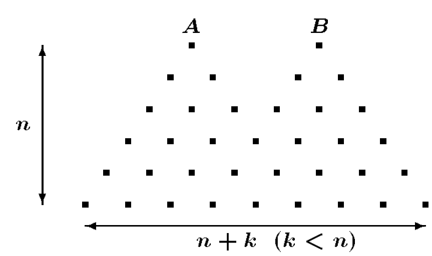
$$F_{2n} \mid \left(F_{3n} + (-1)^n F_n\right) .$$

2611. *Proposed by Michel Bataille, Rouen, France.*

Let O, H and R denote the circumcentre, the orthocentre and the circumradius of triangle ABC, and let Γ be the circle with centre O and radius $OH = \rho$. The tangents to Γ at its points of intersection with the rays $[OA)$, $[OB)$ and $[OC)$ form a triangle. Find the circumradius of this triangle as a function of R and ρ.

2612*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Two "Galton"–figures are given as follows:



(There are n levels in total; there are k levels such that there is no "intersection" between the levels emanating from A and B.)

Let two balls start at the same time from A and B. Each ball moves either ↙ or ↘ with probability $\frac{1}{2}$.

Determine the probability $P(n,k)$ ($1 \leq k < n$) such that the two balls reach the bottom level without colliding.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The name of Walther Janous, Ursulinengymnasium, Innsbruck, Austria was inadvertently omitted from the list of solvers of 2495.

2490. [1999 : 505, 2000 : 517] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $\alpha > 1$. Denote by x_n the only positive root of the equation:

$$(x+n^2)(2x+n^2)(3x+n^2)\cdots(nx+n^2) \;=\; \alpha\, n^{2n}\,.$$

Find $\lim\limits_{n\to\infty} x_n$.

Comment by Nikolaos Dergiades, Thessaloniki, Greece.

In the comments after the solutions, it was stated that:

Konečný gave a one-line "proof" based on the "fact" that

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{kx}{n^2}\right) \;=\; e^{x/2}$$

which he believed "must be well known", but could not find a reference.

Now, it is well known that if $\lim\limits_{n\to\infty} a_n = 0$, then $\lim\limits_{n\to\infty}\dfrac{\ln(1+a_n)}{a_n} = 1$, and since $(1+a_n)^{b_n} = e^{a_n b_n \frac{\ln(1+a_n)}{a_n}}$, we have

$$\lim_{n\to\infty}(1+a_n)^{b_n} \;=\; e^{\lim\limits_{n\to\infty}(a_n b_n)}\,. \qquad (1)$$

Using the generalized Bernoulli's Inequality, we have, for $1 \le k \le n$,

$$\left(1+\frac{x}{n}\right)^{\frac{k}{n}} \;\le\; \left(1+\frac{kx}{n^2}\right) \;\le\; \left(1+\frac{x}{n^2}\right)^{k}, \qquad \text{or}$$

$$\left(1+\frac{x}{n}\right)^{\frac{n+1}{2}} \;\le\; \prod_{k=1}^{n}\left(1+\frac{kx}{n^2}\right) \;\le\; \left(1+\frac{x}{n^2}\right)^{\frac{n(n+1)}{2}}\,.$$

From (1), we have that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n+1}{2}} \;=\; e^{\lim\limits_{n\to\infty}\left(\frac{x(n+1)}{2n}\right)} \;=\; e^{x/2} \qquad \text{and}$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n^2}\right)^{\frac{n(n+1)}{2}} \;=\; e^{\lim\limits_{n\to\infty}\left(\frac{xn(n+1)}{2n^2}\right)} \;=\; e^{x/2}\,,$$

giving that $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\left(1+\dfrac{kx}{n^2}\right) \;=\; e^{x/2}$.

2501. [2000 : 45] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle BAC$ and $\angle ABC$ meet BC and AC at D and E respectively. Suppose that $AB + BD = AE + EB$. Characterize $\triangle ABC$.

I. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

On the extension of AB we take $BF = BD$. [Because BE bisects $\angle ABD$, which is an external angle of $\triangle BFD$], $\angle BFD = \angle BDF = \angle ABE = \angle EBD$. If F' on AC is symmetric with F about the bisector AD, then

$$\angle EF'D \;=\; \angle EBD\,. \qquad (1)$$

We now have $AB + BD = AE + EB$ is equivalent to $AE + EB = AF = AF' = AE + EF'$, or

$$EB \;=\; EF'\,. \qquad (2)$$

If the points B, D, F' are collinear, then F' coincides with C and from (1) we have

$$\angle ABC \;=\; 2\angle ACB\,.$$

If the points B, D, F' are not collinear, then since $\triangle EBF'$ is isosceles (from (2)) we conclude that $\triangle BDF'$ is also isosceles; that is, $BD = DF' = DF$. Hence, $\triangle BFD$ is equilateral and

$$\angle ABC \;=\; 120^\circ\,.$$

[*Comment*. Although Dergiades does not say so explicitly, it is clear that his argument is reversible, so that, conversely, if ABC is a triangle with either $\angle ABC = 2\angle ACB$ or $\angle ABC = 120^\circ$, then $AB + BD = AE + EB$.]

II. *Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

Since $BD = \frac{ac}{b+c}$, $AE = \frac{bc}{a+c}$, and $EB = \frac{2ac}{a+c}\cos\left(\frac{B}{2}\right)$, the given condition $AB + BD = AE + EB$ is equivalent to

$$c + \frac{ac}{b+c} \;=\; \frac{bc}{a+c} + \frac{2ac}{a+c}\cos\left(\tfrac{B}{2}\right)\,,$$

which simplifies to

$$\frac{a^2 - b^2 + c^2 + 2ac + ab}{b+c} \;=\; 2a\cos\left(\tfrac{B}{2}\right)\,.$$

In the last equation we replace $a^2 - b^2 + c^2$ by $2ac\cos B$ and divide both sides by a. The result is

$$\frac{2c(1+\cos B) + b}{b+c} = 2\cos\left(\tfrac{B}{2}\right)\,,$$

or

$$\frac{4c\cos^2\frac{B}{2} + b}{b+c} \;=\; 2\cos\left(\tfrac{B}{2}\right)\,.$$

The roots of this quadratic equation in $\cos\left(\frac{B}{2}\right)$ are $\frac{1}{2}$ and $\frac{b}{2c}$. If $\cos\left(\frac{B}{2}\right) = \frac{1}{2}$, then $B = 120°$. If $\cos\left(\frac{B}{2}\right) = \frac{b}{2c}$, then we use the Law of Sines, $\frac{b}{c} = \frac{\sin B}{\sin C}$, and replace $\sin B$ by $2\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)$ to deduce that $\sin\left(\frac{B}{2}\right) = \sin C$, which is equivalent to $B = 2C$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain (a second solution); MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; and the proposer.

Several solvers used the fact that $ac - b^2 + c^2$ is equivalent to the condition $\angle B = 2\angle C$; Amengual reports that this equivalence also appears in [1976 : 74], [1984 : 287], *and* [1996 : 265-267].

2502. [2000 : 45] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, the internal bisectors of $\angle BAC$, $\angle ABC$ and $\angle BCA$ meet BC, AC and AB at D, E and F respectively. Let p and q be the perimeters of $\triangle ABC$ and $\triangle DEF$ respectively.

Prove that $p \geq 2q$, and that equality holds if and only if $\triangle ABC$ is equilateral.

Solution by Michel Bataille, Rouen, France.

Because AD is the internal bisector of $\angle BAC$, $\dfrac{DB}{DC} = \dfrac{c}{b}$, so that $\dfrac{DB}{c} = \dfrac{DC}{b} = \dfrac{DB+DC}{b+c} = \dfrac{a}{b+c}$, and $DB = \dfrac{ac}{b+c}$. Similarly, $FB = \dfrac{ac}{a+b}$.

The Law of Cosines in $\triangle BDF$ gives $DF^2 = FB^2 + DB^2 - 2FB \cdot DB \cdot \cos B$. Using also $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$ and rearranging [note: $(b+a)^2 + (b+c)^2 = 2(b+a)(b+c) + (a-c)^2$], we obtain:

$$DF^2 = \frac{ac}{(b+a)^2(b+c)^2}\left(b(a+b+c)(b^2 - (a-c)^2) + ab^2c\right).$$

We therefore have $DF^2 \leq \dfrac{ac}{(b+a)^2(b+c)^2}\left(b(a+b+c)b^2 + ab^2c\right)$, or

$$DF \leq b\sqrt{\frac{ac}{(b+a)(b+c)}}. \tag{1}$$

(We note that equality holds if and only if $a = c$.) Using the AM–GM Inequality, we see that

$$\begin{aligned} DF &\le \frac{b\sqrt{ac}}{\sqrt{2\sqrt{ba}\cdot 2\sqrt{bc}}} = \frac{1}{2}\sqrt{b}\cdot\sqrt{\sqrt{ac}} \le \frac{1}{2}\cdot\frac{b+\sqrt{ac}}{2} \\ &\le \frac{1}{4}\left(b+\frac{a+c}{2}\right) = \frac{b}{4}+\frac{a}{8}+\frac{c}{8}. \end{aligned}$$

Analogous inequalities can be obtained for FE and ED leading to:

$$\begin{aligned} q &= DF+FE+ED \\ &\le \left(\frac{b}{4}+\frac{a}{8}+\frac{c}{8}\right)+\left(\frac{a}{4}+\frac{b}{8}+\frac{c}{8}\right)+\left(\frac{c}{4}+\frac{a}{8}+\frac{b}{8}\right) = \frac{p}{2}. \end{aligned}$$

Thus $p \ge 2q$ as desired.

If $\triangle ABC$ is not equilateral, say $a \ne c$, inequality (1) is strict (as noted) and $p > 2q$. If $\triangle ABC$ is equilateral, $\triangle DEF$ is its median triangle, so that $p = 2q$. This completes the proof.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

We received two generalization, one from Janous and one from Woo. Janous proves,

For a triangle with semiperimeter s, if its internal angle bisectors meet the opposite sides in the vertices of a triangle whose side lengths are a_1, b_1, c_1, then

$$a_1^2+b_1^2+c_1^2 \le \frac{s^2}{3},$$

with equality if an only if the initial triangle is equilateral.

Unfortunately his proof requires a computer to manipulate the unwieldy formulas he obtains. By applying the inequality between the arithmetic and square-root means to his inequality he gets $a_1+b_1+c_1 \le s$, which is the inequality of Seimiya.

Woo's result seems to require that the original triangle be acute, namely:

Let $\triangle ABC$ have $A < B < C < \frac{\pi}{2}$. Let AA', BB', CC' be its medians and let AA'', BB'', CC'' be its altitudes. For each point P inside the triangle let P_1, P_2, P_2, be the points on the sides such that the cevians AP_1, BP_2, CP_3 concur at P. Then $\triangle P_1P_2P_3$ will have perimeter less than half that of $\triangle ABC$ if P is inside the region bounded by the lines AA', AA'', CC', CC''.

Semiya's inequality (restricted to acute triangles) therefore follows from Woo's because the angle bisector from a vertex lies between the median and altitude from that vertex.

2503. [2000 : 45] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The incircle of $\triangle ABC$ touches BC at D, and the excircle opposite to B touches BC at E. Suppose that $AD = AE$. Prove that

$$2\angle BCA - \angle ABC = 180^\circ .$$

Solution by Nikolaos Dergiades, Thessaloniki, Greece; and Gerry Leversha, St. Paul's School, London, England.

Let a, b, c be the side lengths and A, B, C be the angles of $\triangle ABC$. If s is the semiperimeter, then we know that $BD = s - b$, $BE = s$ and using the Cosine Rule with triangles ABD and ABE, we obtain that

$$\begin{aligned} AD^2 &= c^2 + (s-b)^2 - 2c(s-b)\cos B , \\ AE^2 &= c^2 + s^2 - 2cs\cos B , \end{aligned}$$

and since $AD = AE$, on subtracting, we get

$$-2sb + b^2 + 2cb\cos B = 0$$

or

$$-(a+b+c) + b + 2c\cos B = 0$$

or

$$2c\cos B = a + c .$$

The last equality can be rewritten, using the Sine Law, as

$$2\sin C\cos B = \sin(B+C) + \sin C$$

or

$$\sin(C - B) = \sin C .$$

Since $C - B \neq C$, we have

$$C - B = 180^\circ - C ;$$

that is, $2\angle BCA - \angle ABC = 180^\circ$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Cotham School, Bristol, UK; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2504. [2000 : 45] *Proposed by Hayo Ahlburg, Benidorm, Spain, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that A, B and C are the angles of a triangle. Determine the best lower and upper bounds of $\prod_{\text{cyclic}} \cos(B-C)$.

Solution by Kee-Wai Lau, Hong Kong.

The answer is

$$-\frac{1}{8} \leq \prod_{\text{cyclic}} \cos(B-C) \;\leq\; 1\,.$$

Firstly,

$$\prod_{\text{cyclic}} \cos(B-C) \;\leq\; \left|\prod_{\text{cyclic}} \cos(B-C)\right| \;=\; \prod_{\text{cyclic}} |\cos(B-C)| \;\leq\; 1\,.$$

Secondly, using the identity

$$\cos X \cos Y \;=\; \frac{1}{2}[\cos(X+Y) + \cos(X-Y)]\,,$$

we get

$$\begin{aligned}\prod_{\text{cyclic}} \cos(B-C) &= \frac{1}{2}\Big(\cos(B-A) + \cos(A+B-2C)\Big)\cos(A-B) \\ &= \frac{1}{2}\left(\cos(A-B) + \frac{\cos(A+B-2C)}{2}\right)^2 \\ &\qquad - \frac{1}{8}\cos^2(A+B-2C) \\ &\geq -\frac{1}{8}\cos^2(A+B-2C) \;\geq\; -\frac{1}{8}\,.\end{aligned}$$

The equilateral triangle shows that the upper bound 1 cannot be improved, and the degenerate triangle with $A = 2\pi/3$, $B = \pi/3$, $C = 0$ shows that the lower bound $-1/8$ cannot be improved either.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; HENRY LIU, student, University of Cambridge, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One other reader misread the product for a sum.

Klamkin points out that the result is true for arbitrary *angles A, B, C, and in fact Lau's proof works in that generality.*

Most readers obtained strict inequality in the lower bound, which is correct if degenerate triangles are not allowed.

2506. [2000 : 46] *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

In the lattice plane, determine all lattice straight lines that are tangent to the unit circle. (A lattice straight line is a straight line containing two lattice points.)

Solution by Skidmore College Problem Group, Skidmore College, Saratoga Springs, New York.

Theorem. $ax + by = c$ is the equation of a lattice straight line tangent to the unit circle if and only if $a^2 + b^2 = c^2$, where a, b, $c \in \mathbb{Z}$.

Proof: ($\Longleftarrow$) Let S^1 be the unit circle, and suppose that a, b, $c \in \mathbb{Z}$ with $a^2 + b^2 = c^2$. Let ℓ be the line with equation $ax + by = c$. It is a standard result from number theory that, since $\gcd(a, b)|c$, the Diophantine equation $ax + by = c$ has infinitely many integral solutions (compare, for example, *Elementary Number Theory* by Underwood Dudley, 2nd Edition, p. 25, Theorem 1). Thus, ℓ is a lattice straight line. Also, $C(a/c, b/c) \in S^1 \cap \ell$. Moreover, $m_\ell = -a/b$ and $m_{OC} = b/a$, so that $\ell \perp OC$, implying that ℓ is tangent to S^1.

($\Longrightarrow$) Suppose that ℓ is a lattice straight line through the points $A(r, s)$ and $B(t, u)$, where r, s, t, $u \in \mathbb{Z}$, and suppose that ℓ is tangent to S^1. Then ℓ has equation $ax + by = c$, where $a = s - u$, $b = t - r$, and $c = st - ru$. Since ℓ is tangent to S^1, we know that the distance from ℓ to the origin is 1, so that, using the formula for the distance between a point and a line in the coordinate plane, we get

$$\frac{|a(0) + b(0) - c|}{\sqrt{a^2 + b^2}} = 1.$$

Simple algebra then yields $a^2 + b^2 = c^2$.

Note: It is easy to see that this result generalizes to Euclidean 3–space $\mathbb{R}^3$. First, we say that a plane π is a *lattice plane* if π contains three non-collinear lattice points. If S^2 is the unit sphere, then a lattice plane π is tangent to S^2 if and only if π has equation $ax + by + cz = d$, where a, b, c, $d \in \mathbb{Z}$ and $a^2 + b^2 + c^2 = d^2$. With appropriate definitions, this result can be further generalized in the obvious way to $\mathbb{R}^n$, using the unit hypersphere S^{n-1} and lattice $(n - 1)$–dimensional subspaces.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, graduate student, University of Cambridge, Cambridge, UK; DAVID LOEFFLER, student, Cotham School, Bristol, UK; and the proposer.

Bradley, uses t to parameterize the unit circle:

$$x = \frac{1 - t^2}{1 + t^2}, \qquad y = \frac{2t}{1 + t^2}$$

which gives him the equation of the tangent at 't' as

$$x(1 - t^2) + 2ty = 1 + t^2.$$

He remarks that resolving this with the knowledge that t is rational (which he proves is the case for this problem) shows that for each such pair (x, y), there exists an integer z such that $x^2 + y^2 = 1 + z^2$. He has written a paper for the Mathematical Gazette (Note 80.34, pp. 49-51) relating all solutions of the Diophantine equation $x^2 + y^2 = 1 + z^2$ to Pythagorean triplets and conversely.

2507. [2000 : 46] *Proposed by Ice B. Risteski, Skopje, Macedonia.*

Show that there are infinitely many pairs of distinct natural numbers, n and k such that $\gcd(n! + 1, k! + 1) > 1$;

I. *Solution by the Austrian IMO Team-2000; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

Let k be any positive integer such that $k + 1$ is not a prime, and let p be a prime such that $p \mid k! + 1$. Then $p > k$ and thus $p \geq k + 2$ for otherwise $p = k + 1$, a contradiction. Let $n = p - 1$. Then $n > k$. Since $p \mid n! + 1$ by Wilson's Theorem, we have $\gcd(n! + 1, k! + 1) \geq p > 1$.

II. *Solution by Kee-Wai Lau, Hong Kong; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $k \geq 3$ be any *odd* integer and p be a prime divisor of $k! + 1$. Then $p > k$ implies $p \geq k + 2$ as both p and k are odd. Let $n = p - 1$. Then $n > k$ and the rest follows as in I above.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHARLES DIMINNIE and TREY SMITH, Angelo State University, San Angelo, TX, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, graduate student, University of Cambridge, Cambridge, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There was one partially incorrect solution submitted.

Janous remarked that a solution would follow immediately from the following known theorem which can be found in Problems in Number Theory *(in Bulgarian), Sofia, 1985 by St. M. Dodvnekov and K.B. Chakarian.*

> ***Theorem**: There exist infinitely many primes p with the property that there is a unique natural number $q < p$ such that $p \mid (q-1)! + 1$. For the present problem, simply take $n = p - 1$ and $k = q - 1$.*

2508. [2000 : 46] *Proposed by J. Chris Fisher, University of Regina, Regina, Saskatchewan.* (Corrected) In problem 2408 [1999 : 49; 2000 : 55] we defined a point P to be *Cevic* with respect to $\triangle ABC$ if the vertices D, E, F of its pedal triangle determine concurrent cevians; more precisely, D, E, F are the feet of perpendiculars from P to the respective sides BC, CA, AB, while AD, BE, CF are concurrent.

1. Show that a point D on the line BC can determine 0, 1, 2, or infinitely many positions for E on AC for which P is Cevic.

2. Describe the possible locations of E if D divides the segment BC in the ratio $\lambda : 1 - \lambda$ (when P is Cevic and λ is an arbitrary real number).

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA (*abridged by the editor*).

1. When D is the mid-point of BC and $AB = AC$, then [because the figure is symmetric about the line AD] every point P on AD would be Cevic, and so there would be infinitely many positions for E. We now prove that *when D is not the mid-point of BC there can be at most two positions for E*. Assume that $BD < DC$. [We shall see that when D is the mid-point of BC and $AC \neq AB$, there is always a unique position for E on AC; the argument that follows shows that the second position of E moves out to infinity as D approaches the mid-point.] Let D' be the point on line BC such that $BD : DC = BD' : D'C$, with B between D and D'; D' is called the *harmonic conjugate of D with respect to B and C*. Let H, K be points on lines AB, AC such that $AHD'K$ is a parallelogram. Let O be the point for which $OH \perp HA$ and $OK \perp KA$. If E, F are variable points on AC, AB such that AD, BE, CF are concurrent, we shall call $\triangle DEF$ a *cevian* triangle. For any point E on AC, $\triangle DEF$ cevian implies that D' lies on EF. [Indeed, this is a consequence of Ceva's Theorem; see, for example, Roger A. Johnson, *Advanced Euclidean Geometry*, Theorem 220 on p. 149.] As the line EF moves about D', define P to be the point such that $PE \perp AC$ and $PF \perp AB$. We assert that the locus of P is the hyperbola Z through A with OH and OK as asymptotes:

Let AP cut OH at Q, and OK at R. Let A', P' be points on OH and A'', P'' be points on OK such that $OA'AA''$ and $OP'PP''$ are parallelograms. Then $\frac{AA''}{PP''} = \frac{RA}{RP} = \frac{KA}{KE} = \frac{D'F}{D'E} = \frac{FH}{AH} = \frac{PQ}{AQ} = \frac{PP'}{AA'}$. Hence, the parallelograms $OA'AA''$ and $OP'PP''$ have equal areas, which implies the assertion that P lies on the hyperbola Z through A with OK and OH as asymptotes.

Finally, it is exactly when $DP \perp BC$ that the cevian $\triangle DEF$ is also the pedal triangle of P. The line through D that is perpendicular to BC will intersect Z in at most two points, which proves 1.

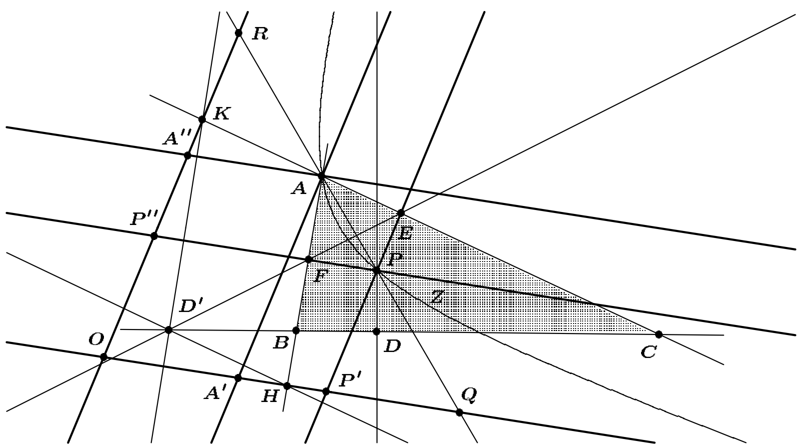
Editor's comment. Bataille presents the same argument from a projective point of view; it is very concise: Start with a point D on BC. As in Woo's argument, for each E on AC one determines the cevian triangle DEF, from which P is defined as the intersection point of the perpendiculars to the sides from E and F; define D^* to be the foot of perpendicular from P to BC. The mapping that takes D to D^* is a composition of perspectivities, and is thus a projectivity of line BC. Consequently, unless it is the identity map, it has at most two invariant (= double) points; at those positions where $D = D^*$ the resulting point P is Cevic. We return to Woo and his treatment of part 2.

2. We will now see how to construct for a given point D on BC, those points E and E', F and F' for which the corresponding point P is Cevic (when such points exist). First, we solve a general construction problem: Given $\triangle OXY$ and a point A within $\angle XOY$ but not on XY, *construct points P*

and P' on XY such that line AP cuts lines OX, OY at Q, R with $AQ = PR$, and line AP' cuts lines OX, OY at Q', R' with $AQ' = P'R'$.

Solution. Draw line $X'AY' || XY$, cutting lines OX, OY at X', Y'. Draw $AN \perp X'AY'$, cutting the semicircle on $X'Y'$ as diameter at N. Then $AN = \sqrt{X'A \times AY'}$. Next draw a line parallel to XY at distance AN from it, cutting the semicircle on XY as diameter possibly at two points J, J'. (Of course, J and J' may coincide or fail to exist.) Let P, P' be the projections of J, J' on XY. Let line AP cut lines OX, OY at Q, R. Let line AP' cut lines OX, OY at Q', R'. We now prove $AQ = PR$: Since $JP = AN$, $X'A \times AY' = XP \times PY$. Hence $\dfrac{QA}{QP} = \dfrac{X'A}{XP} = \dfrac{PY}{AY'} = \dfrac{RP}{RA}$. Therefore $QA = PR$. Similarly, $Q'A = P'R'$. However, if $X'A \times AY'$ is too large, making $AN > \dfrac{XY}{2}$, then J, J' and P, P' do not exist. This ends the construction.

Returning to the main problem, we refer to the diagram.



Given A, B, C, D we can construct D' and quadrilateral $OHAK$. We can then construct up to two possible positions of P lying on the line through D perpendicular to BC such that AP cuts OH at Q and OK at R, with $PQ = AR$. Then $\dfrac{A''A}{P''P} = \dfrac{RA}{RP} = \dfrac{PQ}{AQ} = \dfrac{PP'}{AA'}$, which proves that P lies on the hyperbola. Then DEF is the pedal triangle of P as well as a cevian triangle of $\triangle ABC$, and P is Cevic as desired. However, for some positions of D it is possible that P cannot be constructed, in which case no Cevic point exists.

Editor's comment. Most solvers used algebra to obtain a quadratic equation for $\mu = \dfrac{CE}{CA}$, whose zeros give the positions of E for which P is Cevic. Dergiades wrote his equation in terms of $\lambda = \dfrac{BD}{BC}$ and the sides a, b,

c of $\triangle ABC$:

$$p\mu^2 + q\mu + r \;=\; 0\,,$$

where $\begin{cases} p = 2b^2 - 4b^2\lambda \\ q = -a^2 - 3b^2 + c^2 + 4(a^2+b^2)\lambda - 4a^2\lambda^2 \\ r = a^2 + b^2 - c^2 + (-3a^2 - b^2 + c^2)\lambda + 2a^2\lambda^2 \end{cases}$.

He gave explicit examples of the various possibilities using the right triangle whose sides are $a = 6$, $b = \sqrt{15}$, $c = \sqrt{21}$. The reader can check for himself that when

$\lambda = \frac{1}{3}$, $10\mu^2 - 8\mu + 4 = 0$
has no real solution, and there can be *no* Cevic point.

$\lambda = \frac{1}{2}$, $6\mu - 3 = 0$
has one real solution, and there is a *unique* Cevic point.

$\lambda = \frac{7}{16}$, $120\mu^2 + 54\mu - 27 = 0$
has two real solutions, and there are *two* Cevic points.

Also solved by MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; and the proposer.

Indeed, the proposer's solution finally arrived with an apology, a correction, and a formula that purports to give the probability that he gets the theorem wrong on the first try. Alas, the formula can take values greater than 1.

2509. [2000 : 46] *Proposed by Ice B. Risteski, Skopje, Macedonia.*

Show that there are infinitely many pairs of distinct natural numbers, n and k such that $\gcd(n! - 1, k! - 1) > 1$.

Solution by Charles Diminnie and Trey Smith, Angelo State University, San Angelo, TX, USA.

First note that if p is a prime, then by Wilson's Theorem we have $(p-1)(p-2)! = (p-1)! \equiv -1 \equiv p-1 \pmod{p}$ and hence, $(p-2)! \equiv 1 \pmod{p}$, since $\gcd(p, p-1) = 1$.

[Ed: This is actually well known and in fact is the result usually established first in the proof of Wilson's Theorem.]

Let $k > 3$ be any even integer. Then $k! - 1 > 1$ and $k + 2$ is not a prime. Let p be any prime divisor of $k! - 1$. Then $k \neq p - 2$ and $\gcd((p-2)! - 1, k! - 1) \geq p > 1$.

Also solved by the AUSTRIAN IMO TEAM 2000; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, graduate student, University of Cambridge, Cambridge, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; KENNETH M. WILKE, Topeka, KS, USA; and the proposer.

2510. [2000 : 46] *Proposed by Ho-joo Lee, student, Kwangwoon University, South Korea.*

In $\triangle ABC$, $\angle ABC = \angle ACB = 80°$ and P is on the line segment AB such that $AP = BC$. Find $\angle BPC$.

I. *Solution by Geoffrey A. Kandall, Hamden, CT, USA.*

Let $AP = BC = r$, $PC = s$ and $\angle BPC = \theta$. We have $\angle PAC = 20°$ and $\angle ACP = \theta - 20°$. Using the Law of Sines twice, we have

$$\frac{\sin(\theta - 20°)}{\sin(20°)} = \frac{r}{s} = \frac{\sin(\theta)}{\sin(80°)}.$$

Hence,

$$\frac{\sin(\theta - 20°)}{\sin(\theta)} = \frac{\sin(20°)}{\cos(10°)}.$$

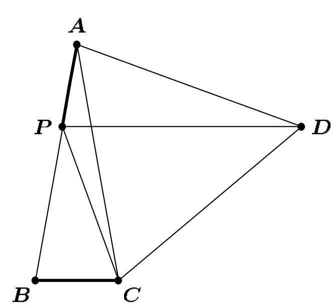
It is easy to see that $\theta = 30°$ is a solution of this equation. It is the only solution since the function

$$f(\theta) = \frac{\sin(\theta - 20°)}{\sin(\theta)} = \cos(20°) - \sin(20°)\cot(\theta)$$

is strictly increasing for $0° < \theta < 180°$.

II. Solution by Jens Windelband, Hegel-Gymnasium, Magdeburg, Germany and Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.

We draw a point D such that $\triangle DAP$ is congruent to given $\triangle ABC$. Since $\measuredangle BAC = 20\,°$ and $\measuredangle DAP = \measuredangle ABC = 80\,°$, we find $\measuredangle DAC = 60\,°$.



Moreover, $DA = AC$, from which we conclude that $\triangle DAC$ is an isosceles triangle with an included angle of $60\,°$; that is, it is equilateral. Hence points A, P, C lie on a circle with centre D. For chord AP of this circle we find by the central angle theorem $\measuredangle ADP = 20\,° = 2\measuredangle ACP$, or $\measuredangle ACP = 10\,°$. Finally, in $\triangle ACP$ the sum of the measures of the two non-adjacent interior angles $\measuredangle PAC + \measuredangle ACP = 30\,°$ equals the measure of the exterior angle $\measuredangle BPC = 30\,°$.

Also solved (using trigonometry) by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AUSTRIAN IMO–TEAM 2000; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RICHARD B. EDEN, Ateneo de Manila University, Manila, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; DAVID LOEFFLER, student, Cotham School, Bristol, UK; HENRY J. PAN; student, East York C.I., Toronto, Ontario; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; JONATHAN STOREY, student, Nottingham

High School, Nottingham, UK; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; (using only geometry) by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; HENRY LIU, graduate student, University of Cambridge, Cambridge, UK; KEE-WAI LAU, Hong Kong, China; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; ACHILLEAS SINEFAKOPOULOS, student, University of Athens, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and (two solutions, both methods) by the proposer. There was one incomplete solution.

Benito and Frenández made use of an eight sided polygon, which has all sorts of nice properties. Perz and Sinefakopoulous pointed out that this problem is known, being essentially problem 2 of the A-level Junior paper of the Autumn round of the 1991 Tournament of the Towns. There is a solution on page 123, by Andy Liu, in International Mathematics Tournament of the Towns, *1989–1993, edited by P.J. Taylor, Australian Mathematics Trust, 1994.*

2511. [2000 : 46] *Proposed by Ho-joo Lee, student, Kwangwoon University, South Korea.*

In $\triangle ABC$, $\angle ABC = 60^\circ$ and $\angle ACB = 70^\circ$. Point D is on the line segment BC such that $\angle BAD = 20^\circ$. Prove that $AB + BD = AD + DC$.

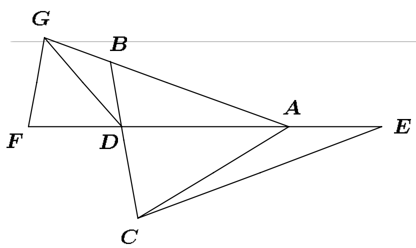
I. *Solution by David Loeffler, student, Cotham School, Bristol, UK.*

Let us scale the triangle so that $AB = 1$. Clearly, $\angle BDA = 100^\circ$. Thus, using the Sine Law on $\triangle ABD$, we have $BD = \dfrac{\sin(20^\circ)}{\sin(100^\circ)}$, and $AD = \dfrac{\sin(60^\circ)}{\sin(100^\circ)}$. A further application of the Sine Law to $\triangle ACD$ gives $DC = AD\dfrac{\sin(30^\circ)}{\sin(70^\circ)} = \dfrac{\sin(60^\circ)}{\sin(100^\circ)}\left(\dfrac{\sin(30^\circ)}{\sin(70^\circ)}\right)$. Now, we have

$$\begin{aligned}
AB + BD &= 1 + \frac{\sin(20^\circ)}{\sin(100^\circ)} = \frac{\sin(70^\circ)(\sin(100^\circ) + \sin(20^\circ))}{\sin(70^\circ)\sin(100^\circ)} \\
&= \frac{2\sin(70^\circ)\sin(60^\circ)\cos(40^\circ)}{\sin(70^\circ)\sin(100^\circ)} \\
&= \frac{2\cos(20^\circ)\sin(60^\circ)\sin(50^\circ)}{\sin(70^\circ)\sin(100^\circ)} \\
&= \frac{\sin(60^\circ)(2\sin(50^\circ)\cos(20^\circ))}{\sin(70^\circ)\sin(100^\circ)} \\
&= \frac{\sin(60^\circ)(\sin(70^\circ) + \sin(30^\circ))}{\sin(70^\circ)\sin(100^\circ)} \\
&= \frac{\sin(60^\circ)}{\sin(100^\circ)}\left(1 + \frac{\sin(30^\circ)}{\sin(70^\circ)}\right) = AD + DC
\end{aligned}$$

as required.

II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*



On the extension of AD, we take $AE = DF = DC$, and on AB, we take $AG = AF$.

Since $\angle DAC = 30^\circ$ and $\angle ADC = 80^\circ$ from problem 2510, we conclude that $\triangle EDC$ is isosceles, with $\angle DEC = \angle GAF = 20^\circ$, which means that $\triangle EDC$ is congruent to $\triangle AGF$.

Hence, $GF = DC = DF$, and further, $\angle GDF = 50^\circ$, so that $\angle BDG = 30^\circ$. Thus, $\angle BGD = 30^\circ$, so that $BG = BD$.

Thus, $AB + BD = AG = AF = AD + DC$.

Also solved by AUSTRIAN IMO–TEAM 2000; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, CT, USA; RICHARD B. EDEN, Ateneo de Manila University, Manila, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong, China; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, graduate student, University of Cambridge, Cambridge, UK; HENRY J. PAN, student, East York C.I., Toronto, Ontario; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria (two solutions, one †); HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; JONATHAN STOREY, student, Nottingham High School, Nottingham, UK; PARAGIOU THEOKLITOS, Limassol, Cyprus, Greece; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESÚS VILLAR RUBIO, Santander, Spain; ALBERT WHITE, St. Bonaventure University, St. Bonaventure, NY, USA; KENNETH M. WILKE, Topeka, KS, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA †; JEREMY YOUNG, student, University of Cambridge, Cambridge, UK; and the proposer. All solvers, except those marked † used trigonometry.

Woo commented that "elegance means avoiding trigonometry".