THE SKOLIAD CORNER

No. 33

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As a contest this issue we give the Senior High School Mathematics Contest, Preliminary Round 1998 of the British Columbia Colleges. My thanks go to the organizer, Jim Totten, The University College of the Cariboo, for forwarding me the contest materials. Time allowed is 45 minutes!

BRITISH COLUMBIA COLLEGES
Senior High School Mathematics Contest
Preliminary Round 1998

Time: 45 minutes

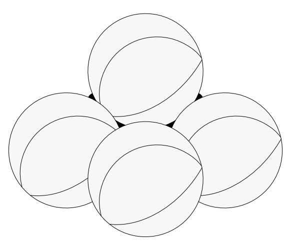
1. The integer $1998 = (n-1)n^n(10n+c)$ where n and c are positive integers. It follows that c equals:

(a) 2 **(b)** 5 **(c)** 6 **(d)** 7 **(e)** 8

2. The value of the sum $\log\frac{1}{2} + \log\frac{2}{3} + \log\frac{3}{4} + \cdots + \log\frac{9}{10}$ is:

(a) -1 **(b)** 0 **(c)** 1 **(d)** 2 **(e)** 3

3. Four basketballs are placed on the gym floor in the form of a square with each basketball touching two others. A fifth basketball is placed on top of the other four so that it touches all four of the other balls, as shown. If the diameter of a basketball is 25 cm, the height, in centimetres, of the centre of the fifth basketball above the gym floor is:



(a) $25\sqrt{2}$ **(b)** $\frac{25}{2}\sqrt{2}$ **(c)** 20 **(d)** $\frac{25}{2}(1+\sqrt{2})$ **(e)** $25(1+\sqrt{2})$

4. Last summer, I planted two trees in my yard. The first tree came in a fairly small pot, and the hole that I dug to plant it in filled one wheelbarrow load of dirt. The second tree came in a pot, the same shape as that of the first tree, that was one-and-a-third times as deep as the first pot and one-and-a-half times as big around. Let us make the following assumptions:

i) The hole for the second tree was the same shape as for the first tree.

ii) The ratios of the dimensions of the second hole to those of the first hole are the same as the ratios of the dimensions of the pots.

Based on these assumptions, the number of wheelbarrows of dirt that I filled when I dug the hole for the second tree was:

(a) 2 (b) 2.5 (c) 3 (d) 3.5 (e) none of these

5. You have an unlimited supply of 5-gram and 8-gram weights that may be used in a pan balance. If you use only these weights and place them only in one pan, the largest number of grams that you cannot weigh is:

(a) 22 (b) 27 (c) 36 (d) 41 (e) there is no largest number of grams

6. If all the whole numbers from 1 to 1,000,000 are printed, the number of times that the digit 5 appears is:

(a) 100,000 (b) 500,000 (c) 600,000 (d) 1,000,000 (e) 2,000,000

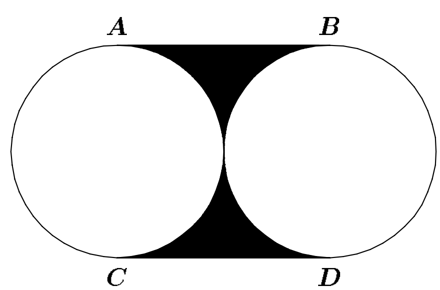
7. The perimeter of a rectangle is x centimetres. If the ratio of two adjacent sides is $a : b$, with $a > b$, then the length of the shorter side, in centimetres, is:

(a) $\frac{bx}{a+b}$ (b) $\frac{x}{2} - b$ (c) $\frac{2bx}{a+b}$ (d) $\frac{ax}{2(a+b)}$ (e) $\frac{bx}{2(a+b)}$

8. The sum of the positive solutions to the equation $x^{x\sqrt{x}} = (x\sqrt{x})^x$ is:

(a) 1 (b) $1\frac{1}{2}$ (c) $2\frac{1}{4}$ (d) $2\frac{1}{2}$ (e) $3\frac{1}{4}$

9. Two circles, each with a radius of one unit, touch as shown. AB and CD are tangent to each circle. The area, in square units, of the shaded region is:



(a) π (b) $\frac{\pi}{4}$ (c) $2 - \frac{\pi}{2}$ (d) $4 - \pi$ (e) none of these

10. A parabola with a vertical axis of symmetry has its vertex at $(0, 8)$ and an x–intercept of 2. If the parabola goes through $(1, a)$, then a is:

(a) 5 (b) 5.5 (c) 6 (d) 6.5 (e) 7

11. A five litre container is filled with pure orange juice. Two litres of juice are removed and the container is filled up with pure water and mixed thoroughly. Then two litres of the mixture are removed and again the container is filled up with pure water. The percentage of the final mixture that is orange juice is:

(a) 27 (b) 25 (c) 30 (d) 36 (e) 24

12. The lengths of the sides of a triangle are $b + 1$, $7 - b$ and $4b - 2$. The number of values of b for which the triangle is isosceles is:

(a) 0 (b) 1 (c) 2 (d) 3 (e) none of these

13. The number of times in one day when the hands of a clock form a right angle is:

(a) 46 (b) 22 (c) 24 (d) 44 (e) 48

14. In my town some of the animals are really strange. Ten percent of the dogs think they are cats and ten percent of the cats think they are dogs. All the other animals are perfectly normal. One day, I tested all the cats and dogs in the town and found that 20% of them thought that they were cats. The percentage of the dogs and cats in the town that really are cats is:

(a) 12.5 (b) 18 (c) 20 (d) 22 (e) 22.5

15. A short hallway in a junior high school contains a bank of lockers numbered one to ten. On the last day of school the lockers are emptied and the doors are left open. The next day, a malicious math student walks down the hallway and closes the door of every locker that has an even number. The following day, the same student again walks down the hallway and, for every locker whose number is a multiple of three, closes the door if it is open and opens it if it is closed. On the next day, the student does the same thing with every locker whose number is divisible by four. If the student continues this procedure for a total of nine days, the number of lockers that are closed after the ninth day is:

(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

Last issue we gave the Junior High School Mathematics Contest, Preliminary Round 1998 of the British Columbia Colleges. Here are the "official solutions", which come our way from the organizer, Jim Totten, The University College of the Cariboo.

BRITISH COLUMBIA COLLEGES
Junior High School Mathematics Contest
Preliminary Round 1998

Time: 45 minutes

1. A number is prime if it is greater than one and divisible only by one and itself. The sum of the prime divisors of 1998 is: **(c)**

Solution. We can factor the number 1998 as follows: $1998 = 2 \times 999 = 2 \times 3^2 \times 111 = 2 \times 3^3 \times 37$. Hence, the sum of its prime divisors is $2 + 3 + 37 = 42$.

2. Successive discounts of 10% and 20% are equivalent to a single discount of: **(c)**

Solution. If P denotes the initial price then the new price after deducting the two consecutive discounts is $P(1 - 0.1)(1 - 0.2) = 0.72P$. This gives the total discount of $(1 - 0.72) \times 100\% = 28\%$.

3. Suppose that $\textcircled{A} = A^2$ and $A \square B = A - 2B$. Then the value of $\textcircled{7} \square \textcircled{3}$ is: **(e)**

Solution. According to our definitions, $\textcircled{7} \square \textcircled{3} = 7^2 \square 3^2 = 49 \square 9 = 49 - 2 \times 9 = 31$.

4. The expression that is not equal to the value of the four other expressions listed is: **(d)**

Solution. The values of the five expressions are:

(a) $1^{\sqrt{9}} + 9 - 8 = 2$,
(b) $(1 + 9) \div (-\sqrt{9} + 8) = 2$,
(c) $-1 \times 9 + \sqrt{9} + 8 = 2$,
(d) $(1 - \sqrt{9}) \times (9 - 8) = -2$,
(e) $19 - 9 - 8 = 2$.

Thus,the expression (d) has a different value than the other four expressions.

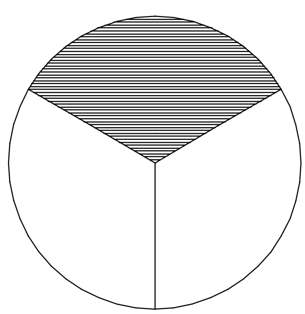
5. The sum of all of the digits of the number $10^{75} - 75$ is: **(c)**

Solution. Consider the procedure for subtracting 75 from 10^{75} "by hand":

$$\begin{array}{r} 100\ldots000 \\ -75 \\ \hline 99\ldots925 \end{array}$$

Hence, the decimal digits of $10^{75} - 75$ are: 5, 2 and seventy-three copies of 9. The sum of the digits is $5 + 2 + 73 \times 9 = 664$.

6. A circle is divided into three equal parts and one part is shaded as in the accompanying diagram. The ratio of the perimeter of the shaded region, including the two radii, to the circumference of the circle is: **(d)**



Solution. The ratio is given by

$$\frac{\frac{2\pi r}{3}+r+r}{2\pi r}=\frac{\frac{2\pi r+6r}{3}}{2\pi r}=\frac{2\pi r+6r}{6\pi r}=\frac{\pi+3}{3\pi}.$$

7. The value of

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}}$$

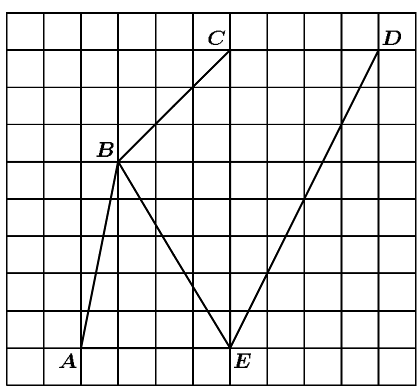
is: **(b)**

Solution. We can simplify this compound fraction by working successively from the bottom to the top of the expression:

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}}=\frac{1}{2-\frac{1}{2-\frac{1}{\left(\frac{3}{2}\right)}}}=\frac{1}{2-\frac{1}{2-\frac{2}{3}}}=\frac{1}{2-\frac{1}{\left(\frac{4}{3}\right)}}=\frac{1}{2-\frac{3}{4}}=\frac{1}{\frac{5}{4}}=\frac{4}{5}.$$

8. If each small square in the accompanying grid is one square centimetre, then the area in square centimetres of the polygon $ABCDE$ is: **(a)**

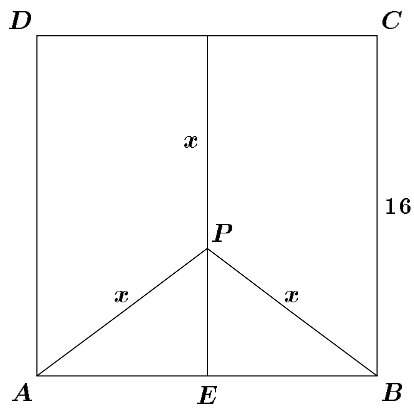
Solution. We can find the area by decomposing the polygon $ABCDE$ into simpler figures, for example, into three triangles: ABE, BCE, and CDE.



If we choose AE, CE, and CD as bases of the triangles then the lengths of the corresponding perpendicular heights are 5, 3, and 8 cm. Hence, the area of the polygon is $\frac{1}{2}\times 4\times 5+\frac{1}{2}\times 8\times 3+\frac{1}{2}\times 4\times 8=38$.

9. A point P is inside a square $ABCD$ whose side length is 16. P is equidistant from two adjacent vertices, A and B, and the side CD opposite these vertices. The distance PA equals: **(e)**

Solution. The situation is illustrated by the following diagram, where x denotes the distance PA.



The Pythagorean Theorem applied to triangle PEB gives $(16-x)^2+8^2 = x^2$, so that $16^2 - 32x + x^2 + 8^2 = x^2$, and $x = \frac{16^2+8^2}{32} = 10$.

10. A group of 20 students has an average mass of 86 kg per person. It is known that 9 people from this group have an average mass of 75 kg per person. The average mass in kilograms per person of the remaining 11 people is: (b)

Solution. If $m_1, m_2, \ldots, m_{20}$ denote the masses of the students then $\frac{m_1+m_2+\cdots+m_{20}}{20} = 86$. Hence, $m_1 + m_2 + \cdots + m_{20} = 20 \times 86 = 1720$. We can assume, without loss of generality, that the average mass of the first nine students is 75; that is, $\frac{m_1+m_2+\cdots+m_9}{9} = 75$. Hence, $m_1 + m_2 + \cdots + m_9 = 9 \times 75 = 675$. The total mass of the remaining 11 people is $1720 - 675 = 1045$. This gives the average of $\frac{1045}{11} = 95$.

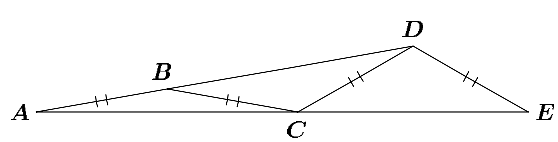
11. In the following display each letter represents a digit:

3	B	C	D	E	8	G	H	I

The sum of any three successive digits is 18. The value of H is: (a)

Solution. We have $3 + B + C = 18$. Consequently, $B + C = 15$. By subtracting this equation from $B + C + D = 18$ we get $D = 3$. Now, $D+E+8 = 18$ gives $E = 10-D = 10-3 = 7$. Finally, $E+8+G = 18$ gives $G = 10 - E = 10 - 7 = 3$, and $8 + G + H = 18$ gives $H = 18 - 8 - 3 = 7$.

12. In the accompanying diagram $\angle ADE = 140°$. The sides are congruent as indicated. The measure of $\angle EAD$ is: (e)

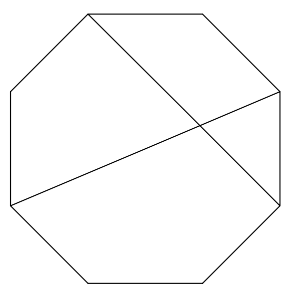


Solution. If $\angle EAD = \alpha$, then also $\angle ACB = \alpha$, since triangle ABC is isosceles. Hence, $\angle CBD = 2\alpha$ as an external angle of triangle ABC. Consequently, $\angle ADC = 2\alpha$, since triangle BCD is isosceles. Further, $\angle ECD = \angle ADC + \angle CAD$ as an external angle of triangle ADC. Hence, $\angle ECD = 2\alpha + \alpha = 3\alpha$. Now, $\angle AED = \angle ECD = 3\alpha$, because triangle CDE is isosceles. This implies that $\angle CDE = 180° - 6\alpha$. Finally, $\angle ADE = \angle ADC + \angle CDE = 2\alpha + 180° - 6\alpha = 180° - 4\alpha$. Thus, $180° - 4\alpha = 140°$. This yields $\alpha = 10°$.

13. The area (in square units) of the triangle bounded by the x–axis and the lines with equations $y = 2x + 4$ and $y = -\frac{2}{3}x + 4$ is: **(e)**

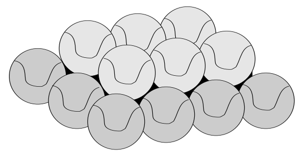
Solution. Two vertices of the triangle lie on the x–axis, so they are the x–intercepts of the lines. The x–intercept of the first line, determined by the equation $2x + 4 = 0$, is -2. Similarly, the second x–intercept, determined by $-\frac{2}{3}x_2 + 4 = 0$, is 6. Consequently, the length of the base of the triangle is $6 - (-2) = 8$. Since both lines have the same y–intercept 4, they intersect each other and the y–axis at level 4 and, consequently, the length of the height of the triangle is 4. Therefore, the area of the triangle is $A = \frac{1}{2}(4 \times 8) = 16$.

14. Two diagonals of a regular octagon are shown in the accompanying diagram. The total number of diagonals possible in a regular octagon is: **(d)**



Solution. Let d_i, for $i = 1, 2, \ldots, 8$, denote the number of diagonals connected to the i^{th} vertex. Then $d_1 = d_2 = \ldots = d_8 = 5$, since each vertex of the octagon is connected to five diagonals. On the other hand, each diagonal joins two vertices. Therefore, in the sum $d_1 + d_2 + \cdots + d_8 = 8 \times 5 = 40$, each diagonal is counted twice. Hence, the number of diagonals in the octagon is 20.

15. A local baseball league is running a contest to raise money to send a team to the provincial championship. To win the contest it is necessary to determine the number of baseballs stacked in the form of a rectangular pyramid. The fifth and sixth levels from the base of the stack of baseballs are shown. If the stack contains a total of seven levels, the number of baseballs in the stack is: **(d)**



Solution. The fifth level has $3 \times 4 = 12$ balls, the sixth $2 \times 3 = 6$ balls, and the seventh $1 \times 2 = 2$ balls. We notice that the number of balls in both sides of the rectangle they form increases by one each time we move one level down. Thus, the total number of balls is $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + 5 \times 6 + 6 \times 7 + 7 \times 8 = 168$.

That completes the *Skoliad Corner* for this issue. We need suitable contests and solutions. I welcome any comments, criticisms, or suggestions for the future direction of this feature.

Advance Announcement

The 1999 Summer Meeting of the Canadian Mathematical Society will take place at Memorial University in St. John's, Newfoundland, from Saturday, 29 May 1999 to Tuesday, 1 June 1999.

The Special Session on Mathematics Education will feature the topic

What Mathematics Competitions do for Mathematics.

The invited speakers are

Ed Barbeau (University of Toronto),
Ron Dunkley (University of Waterloo),
Tony Gardiner (University of Birmingham, UK),
Rita Janes (Newfoundland and Labrador Senior Mathematics League), and
Shannon Sullivan (student, Memorial University).

Requests for further information, or to speak in this session, as well as suggestions for further speakers, should be sent to the session organizers:

Bruce Shawyer and Ed Williams
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