

How the Roots of a Polynomial Vary with its Coefficients: A Local Quantitative Result

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Abstract. A well-known result, due to Ostrowski, states that if $\|P - Q\|_2 < \varepsilon$, then the roots (x_j) of P and (y_j) of Q satisfy $|x_j - y_j| \leq Cn\varepsilon^{1/n}$, where n is the degree of P and Q . Though there are cases where this estimate is sharp, it can still be made more precise in general, in two ways: first by using Bombieri's norm instead of the classical l_1 or l_2 norms, and second by taking into account the multiplicity of each root. For instance, if x is a simple root of P , we show that $|x - y| < C\varepsilon$ instead of $\varepsilon^{1/n}$. The proof uses the properties of Bombieri's scalar product and Walsh Contraction Principle.

1 The General Theory

A well-known result due to Ostrowski [6], [7] can be stated as follows:

(1) Let $P = \sum_0^n a_{n-j}z^j$, $Q = \sum_0^n b_{n-j}z^j$, be two polynomials, satisfying $a_0 = b_0 = 1$, and with respective roots $x_1, \dots, x_n, y_1, \dots, y_n$. Let

$$T = \max\{1, |a_1|, |b_1|, \dots, |a_k|^{1/k}, |b_k|^{1/k}, \dots, |a_n|^{1/n}, |b_n|^{1/n}\}.$$

Then, if the y_j 's are suitably ordered, we have, for all j ,

$$|x_j - y_j| \leq 4nT\delta^{1/n}$$

with

$$\delta = \left(\sum_0^n |a_j - b_j|^2 \right)^{1/2}.$$

(2) Let P, Q be as before; assume moreover that 0 is not a root of P . Assume that, for all j

$$|a_j - b_j| \leq \tau |a_j|$$

where τ is small enough, namely

$$\tau \leq \left(\frac{1}{4n} \right)^n.$$

Then the zeros y_j 's of Q can be ordered in such a way that

$$\left| \frac{y_j}{x_j} - 1 \right| < 8n\tau^{1/n}, \quad \text{for all } j.$$

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Here, we will make this result more precise, in two ways. First, in order to measure $P - Q$, we will use Bombieri's norm, and second, we will take into account the multiplicity of the roots, which, of course, may be different from one to the other: this is why we speak of a "local" result.

Let $P = \sum_0^n a_j z^j$ be a polynomial with complex coefficients and degree n . Its Bombieri's norm is defined by

$$(1) \quad [P] = \left(\sum_0^n \frac{|a_j|^2}{\binom{n}{j}} \right)^{1/2}.$$

This definition is better understood in its original frame, that of homogeneous many-variable polynomials: see Beauzamy-Bombieri-Enflo-Montgomery [3] and Beauzamy-Dégot [4].

Let x_1, \dots, x_n be the roots of P .

Let Q be another polynomial, with same degree, satisfying

$$(2) \quad [P - Q] \leq \varepsilon.$$

Theorem 1 *If x is any zero of P , there exists a zero y of Q , with*

$$(3) \quad |x - y| \leq \frac{n(1 + |x|^2)^{n/2}}{|Q'(x)|} \varepsilon.$$

If ε is small enough, namely

$$(4) \quad \varepsilon \leq \frac{1}{2} \frac{|P'(x)|}{n(1 + |x|^2)^{\frac{n-1}{2}}}$$

then (3) implies

$$(5) \quad |x - y| \leq \frac{2n(1 + |x|^2)^{n/2}}{|P'(x)|} \varepsilon.$$

Before we turn to the proof, let us make some comments about these results.

– Estimates (3) and (5) are homogeneous (which is already an improvement upon Ostrowski's result). Indeed, if all coefficients of P and Q are multiplied by λ , so is ε , and $\varepsilon/|Q'(x)|$ or $\varepsilon/|P'(x)|$ are not modified.

– Theorem 1 is empty if x is not a simple root, either for P or for Q (note that Q can have all roots simple, and P have only one root, as the example of z^n and $z^n + \alpha$ shows).

– The term $(1 + |x|^2)^{1/2}$ can itself be bound by a quantity depending only on the coefficients of the polynomial, for instance by $(1 + R^2)^{1/2}$, where R is the radius of the largest disk, centered at 0, containing all the zeros. An estimate for R can be found in Marden [5]; others may be given, using for instance Mahler's measure of P . Here, we will give later (Theorem 4) a bound depending on Bombieri's norm $[P]$.

Proof of Theorem 1 We need a few simple facts about Bombieri's norm, and the corresponding scalar product, which is just

$$(6) \quad [P, Q] = \sum_{j=0}^n \frac{a_j \bar{b}_j}{\binom{n}{j}},$$

if $P = \sum_{j=0}^n a_j z^j$, $Q = \sum_{j=0}^n b_j z^j$.

Lemma 2 (B. Reznick [8]) For any z_0 ,

$$P(z_0) = [P, (\bar{z}_0 z + 1)^n].$$

(See Reznick [8] or Beauzamy-Dégot [4] for a proof.)

As a consequence, we get

$$(7) \quad |P(z_0)| \leq [P](1 + |z_0|^2)^{n/2}.$$

Indeed

$$|P(z_0)| = |[P, (\bar{z}_0 z + 1)^n]| \leq [P][(\bar{z}_0 z + 1)^n],$$

and an immediate computation shows that

$$[(\alpha z + 1)^n] = (1 + |\alpha|^2)^{n/2}.$$

Another property of the scalar product is

$$(8) \quad [P', R] = n[P, zR]$$

if $\deg P = n$, $\deg R = n - 1$ (see [4] for a proof).

Lemma 3 If $f(z) = az + b$ ($a \neq 0$) satisfies $|f(z_0)| \leq \varepsilon$, there exists z_1 , with $|z_1 - z_0| \leq \varepsilon/|a|$, such that $f(z_1) = 0$. More generally, if $f(z) = a(z - z_1) \cdots (z - z_k)$ satisfies $|f(z_0)| \leq \varepsilon$, one of the roots, say z_1 , satisfies

$$|z_1 - z_0| \leq (\varepsilon/|a|)^{1/k}.$$

Proof of Lemma 3 This is obvious: if

$$|z_0 - z_1| \cdots |z_0 - z_k| \leq \frac{\varepsilon}{|a|},$$

one of the $|z_0 - z_j|$ must be at most equal to $(\varepsilon/|a|)^{1/k}$.

Let us now prove the theorem. Since x is a root of P , we have, by (7):

$$(9) \quad |Q(x)| = |(Q - P)(x)| \leq \varepsilon(1 + |x|^2)^{n/2}.$$

Set

$$\varepsilon' = \varepsilon(1 + |x|^2)^{n/2}.$$

We know by Lemma 2 that

$$(10) \quad Q(x) = [Q, (\bar{x}z + 1)^n].$$

Let us consider

$$f(\zeta) = [Q, (\bar{x}z + 1)^{n-1}(\bar{\zeta}z + 1)],$$

which is an affine function of ζ , satisfying

$$(11) \quad |f(x)| \leq \varepsilon'.$$

By Lemma 3, there is a point x' , $|x' - x| \leq \varepsilon'/|a|$ (where a is the coefficient of ζ in f), such that $f(x') = 0$. Let's compute a . By definition:

$$\begin{aligned} a &= [Q, (\bar{x}z + 1)^{n-1}z] \\ &= \frac{1}{n} [Q', (\bar{x}z + 1)^{n-1}] \quad \text{by (8)} \\ &= \frac{1}{n} Q'(x), \end{aligned}$$

by Lemma 2. So we see that a zero x' of f satisfies

$$(12) \quad |x' - x| \leq \frac{n\varepsilon'}{|Q'(x)|}.$$

Let us now apply Walsh Contraction Principle (Walsh [9], see Beauzamy [1] for a detailed study and proof). Consider

$$(13) \quad \varphi(u_1, \dots, u_n) = [Q, (\bar{u}_1z + 1) \cdots (\bar{u}_nz + 1)].$$

This is a symmetric function of u_1, \dots, u_n , affine with respect to each of them. It satisfies $\varphi(x, \dots, x, x') = 0$. Therefore, in each disk containing both x and x' , and in particular in the disk of diameter xx' , there is a point y such that

$$(14) \quad \varphi(y, \dots, y) = 0.$$

Coming back to the definition of φ , we get

$$\varphi(y, \dots, y) = [Q, (\bar{y}z + 1)^n] = Q(y).$$

So y is a zero of Q . Since it is in the disk of diameter xx' , we have also by (12):

$$|x - y| \leq \frac{n\varepsilon'}{|Q'(x)|},$$

and the first part of Theorem 1 is proved. To get the second part, we write simply:

$$\begin{aligned}
 |P'(x) - Q'(x)| &= |[P' - Q', (\bar{x}z + 1)^{n-1}]| \\
 &= n|[P - Q, z(\bar{x}z + 1)^{n-1}]| \\
 &\leq n[P - Q][z(\bar{x}z + 1)^{n-1}] \\
 &\leq n\varepsilon(1 + |x|^2)^{\frac{n-1}{2}}.
 \end{aligned}$$

So $|Q'(x)| \geq |P'(x)| - n\varepsilon(1 + |x|^2)^{\frac{n-1}{2}}$. If ε is taken as indicated, we get $|Q'(x)| \geq \frac{1}{2}|P'(x)|$; the result follows.

Let us now give a more general version of Theorem 1, valid if x has multiplicity k , empty if it has multiplicity $k + 1$:

Theorem 4 *Let $k \geq 1$ be an integer; P and Q be two polynomials of degree n , with $|P - Q| \leq \varepsilon$. If x is any zero of P , there exists a zero y of Q , with*

$$(15) \quad |x - y| \leq \left(\frac{n!}{(n-k)!} \frac{(1 + |x|^2)^{n/2}}{|Q^{(k)}(x)|} \right)^{1/k} \varepsilon^{1/k}.$$

If ε is small enough, namely

$$(16) \quad \varepsilon \leq \frac{(n-k)!}{2n!} \frac{|P^{(k)}(x)|}{(1 + |x|^2)^{\frac{n-k}{2}}}$$

then (15) implies

$$(17) \quad |x - y| \leq \left(\frac{2n!}{(n-k)!} \frac{(1 + |x|^2)^{n/2}}{|P^{(k)}(x)|} \right)^{1/k} \varepsilon^{1/k}.$$

Proof of Theorem 4 It follows the same lines, so we only indicate the minor changes. We now set

$$(18) \quad f(\zeta) = [Q, (\bar{x}z + 1)^{n-k}(\bar{\zeta}z + 1)^k]$$

which is a polynomial in ζ of degree k , satisfying

$$|f(x)| = |Q(x)| \leq \varepsilon'.$$

By Lemma 3, there is a point x' , with $f(x') = 0$, such that $|x' - x| \leq (\varepsilon'/|a|)^{1/k}$, where a is the coefficient of ζ^k in (18), that is

$$a = [Q, (\bar{x}z + 1)^{n-k}z^k] = \frac{(n-k)!}{n!} Q^{(k)}(x).$$

So we get

$$(19) \quad |x' - x| \leq \left(\frac{n!}{(n-k)!} \frac{\varepsilon'}{|Q^{(k)}(x)|} \right)^{1/k}.$$

Let $\varphi(u_1, \dots, u_n)$ be defined as before. We now get

$$\varphi(\underbrace{x, \dots, x}_{n-k \text{ times}}, \underbrace{x', \dots, x'}_{k \text{ times}}) = 0,$$

so by Walsh's principle, there is a point y , with $\varphi(y, \dots, y) = 0$, satisfying

$$|x - y| \leq \left(\frac{n!}{(n-k)!} \frac{\varepsilon'}{|Q^{(k)}(x)|} \right)^{1/k}.$$

This proves the first part of the Theorem. Now:

$$\begin{aligned} |P^{(k)}(x) - Q^{(k)}(x)| &= |[P^{(k)} - Q^{(k)}, (\bar{x}z + 1)^{n-k}]| \\ &= \frac{n!}{(n-k)!} |[P - Q, z^k(\bar{x}z + 1)^{n-k}]| \\ &\leq \frac{n!}{(n-k)!} \varepsilon (1 + |x|^2)^{\frac{n-k}{2}}, \end{aligned}$$

and the second part follows.

How sharp is the coefficient of ε in estimates (3) or (5)? We do not know exactly, but the order of magnitude is almost best possible. Indeed take $P = z^n - 1$, with $x = 1$, and $Q = z^n + \varepsilon \sqrt{\binom{n}{n/2}} z^{n/2} - 1$ (for n even). Then $[P - Q] = \varepsilon$. The roots of Q are the $n/2$ roots of

$$-\frac{\varepsilon}{2} \sqrt{\binom{n}{n/2}} \pm \sqrt{1 + \frac{\varepsilon^2}{4} \binom{n}{n/2}}$$

and if y is the real zero

$$\left(\sqrt{1 + \frac{\varepsilon^2}{4} \binom{n}{n/2}} - \frac{\varepsilon}{2} \sqrt{\binom{n}{n/2}} \right)^{2/n}.$$

We find

$$|x - y| \sim \frac{\varepsilon}{n} \sqrt{\binom{n}{n/2}} \sim \frac{\varepsilon}{n} 2^{n/2} \left(\frac{2}{\pi n} \right)^{1/4},$$

whereas estimates (3) gave $2^{n/2} \varepsilon$.

2 A Bound for the Largest Zero

We now give an estimate for the largest root of P , in terms of Bombieri's norm. This estimate may be substituted in the term $1 + |x|^2$, in Theorems 1 and 2 above. Of course, now, some normalization is necessary. We choose the usual one, that is $a_n = 1$.

Theorem 5 *If $P = \sum_0^n a_j z^j$ is a polynomial with $a_n = 1$, its roots x_1, \dots, x_n satisfy the estimate*

$$(20) \quad \max_j |x_j| \leq \sqrt{n[P]^2 - 1}.$$

This estimate is best possible.

Proof Let us order the roots so that $|x_1| \geq |x_2| \geq \dots \geq |x_n|$.

Applying Bombieri's inequality (see [2]) to the pair $z - x_1, (z - x_2) \cdots (z - x_n)$ yields:

$$\begin{aligned} [P] &\geq \sqrt{\frac{1!(n-1)!}{n!}} [z - x_1] [(z - x_2) \cdots (z - x_n)] \\ &\geq \frac{1}{\sqrt{n}} (1 + |x_1|^2)^{1/2}, \end{aligned}$$

which gives (20).

The estimate (20) is best possible in the sense that, for every n and every $\varepsilon > 0$, there is a polynomial P which has a root satisfying

$$(21) \quad |x| \geq (1 - \varepsilon) \sqrt{n[P]^2 - 1}.$$

Indeed, with $x > 0$, consider $P = (z - x)(z + \frac{1}{x})^{n-1}$. Since the pair $z - x, (z + \frac{1}{x})^{n-1}$ is extremal for the product (see Beauzamy [2]), we get

$$[P] = \frac{1}{\sqrt{n}} [z - x] \left[z + \frac{1}{x} \right]^{n-1} = \frac{1}{\sqrt{n}} (1 + x^2)^{1/2} \left(1 + \frac{1}{x^2} \right)^{\frac{n-1}{2}},$$

so

$$n[P]^2 - 1 = (1 + x^2) \left(1 + \frac{1}{x^2} \right)^{n-1} - 1,$$

and the inequality

$$x^2 \geq (1 - \varepsilon)^2 \left((1 + x^2) \left(1 + \frac{1}{x^2} \right)^{n-1} - 1 \right),$$

is satisfied, for fixed n and ε , if x is large enough.

3 Blowing Up a Multiple Zero

Theorem 4 indicates that, if you start with a multiple zero x of P , of order k , and if you move P to Q with $[P - Q] \leq \varepsilon$, then x will be moved into y , with $|x - y| \leq C\varepsilon^{1/k}$. But when is such an estimate obtained? Are there cases where a better one holds? The answer is: if the multiple zero stays multiple, stronger estimates can be obtained; the worst case comes if the multiple zero "blows up" into single ones. We will describe this phenomenon in detail in the case of $P = (z - a)^n$.

- Case 1: Q has itself a multiple zero of order n , $Q = (z - b)^n$. Then the condition $[P - Q] \leq \varepsilon$ implies $|b - a| \leq \varepsilon$.

This is clear, from the formula $[P']_{(n-1)} \leq n[P]_{(n)}$, which itself is obtained by elementary manipulations of the binomial coefficients. Here we indicate by a suffix (n) or $(n - 1)$ which norm is used, so as to avoid any confusion.

- Case 2: all roots of Q are simple (or we have no information on Q). Then (17), with $k = n$, gives for $Q = (z - b_1) \cdots (z - b_n)$:

$$(22) \quad |b_j - a| \leq 2^{1/n} (1 + |a|^2)^{1/2} \varepsilon^{1/n}.$$

This estimate is best possible in general: if $Q = (z - a)^n - \varepsilon$, then $[P - Q] = \varepsilon$, and $|b_j - a| = \varepsilon^{1/n}$ for all j .

- Case 3: mixed case $Q = (z - b)^k(z - b_1) \cdots (z - b_{n-k})$. Then, first, the estimate $|b - a| \leq \varepsilon^{1/n}$ can be improved, and we get

$$(23) \quad |b - a| \leq \varepsilon^{1/n-k+1} 2^{1/n-k+1} (1 + |a|^2)^{1/2}.$$

Indeed, we consider $P^{(k-1)}$ and $Q^{(k-1)}$ (which both have a and b respectively as zeros) and apply (22).

Then, also, we can obtain an estimate of the same form for b_1, \dots, b_{n-k} , namely

$$(24) \quad |b_j - a| \leq C(a, n) \varepsilon^{1/n-k+1}, \quad j = 1, \dots, n - k.$$

In order to prove (24), we first assume $a = 0$, that is

$$(25) \quad [z^n - (z - b)^k(z - b_1) \cdots (z - b_{n-k})] \leq \varepsilon,$$

and we know by (23) that

$$(26) \quad |b| = 0(\varepsilon^{1/n-k+1}).$$

We write $\varepsilon' = \varepsilon^{1/n-k+1}$. Let's also write

$$\begin{aligned} z^n - (z - b)^k(z - b_1) \cdots (z - b_{n-k}) &= c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n \\ (z - b_1) \cdots (z - b_{n-k}) &= c'_1 z^{n-k} + c'_2 z^{n-k-1} + \cdots + c'_{n-k}. \end{aligned}$$

Then:

$$|c_1| = |kb + b_1 + \cdots + b_{n-k}| \leq \sqrt{\binom{n}{1}} \varepsilon.$$

Also, we have:

$$\begin{aligned} |c_{j+1}| &= \left| \binom{k}{j+1} b^{j+1} + \binom{k}{j} b^j c'_1 + \cdots + \binom{k}{l} b^l c'_{j-l+1} + \cdots + \binom{k}{1} b c'_j + c'_{j+1} \right| \\ &\leq \sqrt{\binom{n}{j+1}} \varepsilon. \end{aligned}$$

If we assume $|c'_l| = 0(\varepsilon'^l)$, $l = 1, \dots, j$, we deduce from this formula that $|c'_{j+1}| = 0(\varepsilon'^{j+1})$, and so we have shown by induction that

$$(27) \quad |c'_j| = 0(\varepsilon'^j), \quad j = 1, \dots, n - k.$$

We need a lemma.

Lemma 6 Let $R = z^m + a_{m-1}z^{m-1} + \cdots + a_0$ be a polynomial where the coefficients a_{m-1}, \dots, a_0 depend on some parameter α and satisfy

$$|a_{m-1}| = 0(\alpha), |a_{m-1}| = 0(\alpha^2), \dots, |a_{m-k}| = 0(\alpha^k), \dots, a_0 = 0(\alpha^m),$$

when $\alpha \rightarrow 0$. Then all zeros of R are $0(\alpha)$, $\alpha \rightarrow 0$.

This lemma is well-known and follows from estimates found for instance in Marden [5]. Let's give a quick proof. We have $|a_{m-k}| \leq C\alpha^k$, $k = 1, \dots, m$. Let z be a zero of R . Then:

$$1 = -\frac{a_{m-1}}{z} \dots - \frac{a_k}{z^{m-k}} \dots - \frac{a_0}{z^m},$$

and so

$$1 \leq C \sum_1^\infty \left(\frac{|\alpha|}{|z|} \right)^k,$$

which implies $|z| \leq (1 + C)|\alpha|$. So the lemma is proved, and (24) follows from (27).

Let us now consider the general case, $a \neq 0$.

We define $\tau_a P = P(z - a)$. Our estimate will follow from the estimate in the case $a = 0$ and the following.

Lemma 7 For all P, Q , of degree n ,

$$[\tau_a P - \tau_a Q] \leq C(a, n)[P - Q],$$

where

$$C(a, n) = \max_{0 \leq l \leq n} \left\{ \binom{n}{l} (1 + |a|^2)^l \right\}^{1/2}.$$

Proof of Lemma 7 We have

$$\begin{aligned} [\tau_a P]^2 &= \sum_{k=0}^n \frac{1}{\binom{n}{k} k!^2} \left| \sum_{j=0}^{n-k} P^{(k+j)}(0) \frac{a^j}{j!} \right|^2 \\ &\leq \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{(n-k)! |a|^{2j}}{n! k! j!^2} |P^{(k+j)}(0)|^2 \\ &= \sum_{l=0}^n \sum_{j=0}^l \frac{(n-l+j)! l! |a|^{2j}}{(l-j)! j!^2 (n-l)!} \frac{|P^{(l)}(0)|^2}{l^2 \binom{n}{l}} \\ &\leq \left(\max_{0 \leq l \leq n} \sum_{j=0}^l \frac{(n-l+j)! l! |a|^{2j}}{(l-j)! j!^2 (n-l)!} \right) [P]^2. \end{aligned}$$

But

$$\begin{aligned} \sum_{j=0}^l \frac{(n-l+j)! l!}{(l-j)! j!^2 (n-l)!} |a|^{2j} &= \sum_{j=0}^l \binom{n-l+j}{j} \binom{l}{j} |a|^{2j} \\ &\leq \binom{n}{l} \sum_{j=0}^l \binom{l}{j} |a|^{2j} \\ &= \binom{n}{l} (1 + |a|^2)^l, \end{aligned}$$

and the lemma follows.

Remark We do not think that the above constant $C(a, n)$ is sharp. One might think that $(1 + |a|^2)^{n/2}$ is the right constant.

So we see that, starting with $P = (z - a)^n$ and moving it to Q with $[P - Q] \leq \varepsilon$, the estimate $|x - y| \leq \varepsilon^{1/n}$ can always be improved if one of the zeros of Q is multiple. The only case where it is sharp is the case where the multiple zero of P has blown up into n distinct simple zeros for Q .

As we already mentioned in [1], the combination of Bombieri's scalar product and Walsh Contraction Principle provides very efficient tools for the study of quantitative properties of polynomials: the proofs are simpler than the existing ones and the results are sharper. Other results on these lines will be published elsewhere.

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