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NON-UNIQUENESS FOR THE p-HARMONIC FLOW

NORBERT HUNGERBÜHLER

ABSTRACT. If $f_0\colon \Omega \subset \mathbb{R}^m \longrightarrow S^n$ is a weakly p-harmonic map from a bounded smooth domain Ω in $\mathbb{R}^m$ (with $2 < p < m$) into a sphere and if f_0 is not stationary p-harmonic, then there exist infinitely many weak solutions of the p-harmonic flow with initial and boundary data f_0, *i.e.*, there are infinitely many global weak solutions $f\colon \Omega \times \mathbb{R}_+ \longrightarrow S^n$ of

$$\partial_t f - \operatorname{div}(|\nabla f|^{p-2}\nabla f) = |\nabla f|^p f \quad \text{weakly on } \Omega \times \mathbb{R}_+$$
$$f = f_0 \quad \text{on the parabolic boundary of } \Omega \times \mathbb{R}_+.$$

We also show that there exist non-stationary weakly $(m-1)$-harmonic maps $f_0\colon B^m \longrightarrow S^{m-1}$.

1. **Introduction.** Let M and N be compact smooth Riemannian manifolds (M possibly having a boundary) with metrics γ and g respectively. Let m and n denote the dimensions of M and N. For a C^1-map $f\colon M \longrightarrow N$ the p-energy density is defined by

$$e(f)(x) := \frac{1}{p}|df_x|^p \tag{1}$$

and the p-energy by

$$E(f) := \int_M e(f)\,d\mu. \tag{2}$$

Here, p denotes a real number in $[2,\infty[$, $|df_x|$ is the Hilbert-Schmidt norm with respect to γ and g of the differential $df_x \in T_x^*(M)\otimes T_{f(x)}(N)$ and μ is the measure on M which is induced by the metric. In local coordinates $E(f)$ is given by:

$$E_U(f) = \frac{1}{p}\int_\Omega \Big(\gamma^{\alpha\beta}(g_{ij}\circ f)\partial_\alpha f^i \partial_\beta f^j\Big)^{\frac{p}{2}} \sqrt{\gamma}\,dx.$$

Here, $U \subset M$ and $\Omega \subset \mathbb{R}^m$ denote the domain and the range of the coordinates on M and it is assumed that $f(U)$ is contained in the domain of the coordinates chosen on N. Upper indices denote components, whereas ∂_α denotes the derivative with respect to the coordinate variable x^α. We use the usual summation convention.

First we consider variations of the energy-functional of the form $f_\varepsilon = f + \varepsilon\varphi$ with $\varphi \in C_0^\infty\big(B_\rho(x), \mathbb{R}^n\big)$ with $\bar{B}_\rho(x) \subset U$ such that $f_\varepsilon(U)$ is contained in the domain of the coordinates chosen on N provided $|\varepsilon|$ is small enough. The resulting Euler-Lagrange equations are

$$\Delta_p f = -(\gamma^{\alpha\beta} g_{ij}\partial_\alpha f^i \partial_\beta f^j)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\Gamma^l_{ij}\partial_\alpha f^i \partial_\beta f^j \tag{3}$$

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in local coordinates. Here, the operator

$$\Delta_p f := \frac{1}{\sqrt{\gamma}}\partial_\beta\Big(\sqrt{\gamma}(\gamma^{\alpha\beta}g_{ij}\partial_\alpha f^i\partial_\beta f^j)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\partial_\alpha f^l\Big)$$

is called *p-Laplace operator* (for $p = 2$ this is just the Laplace-Beltrami operator and does not depend on N). On the right hand side of (3) the Γ^l_{ij} denote the Christoffel-symbols related to the manifold N. According to Nash's embedding theorem we can think of N as being isometrically embedded in some Euclidean space $\mathbb{R}^k$ since N is compact. Then, if we regard f as a function into $N \subset \mathbb{R}^k$, equation (3) admits a geometric interpretation, namely

$$\Delta_p f \perp T_f N \tag{4}$$

with Δ_p being the p-Laplace operator with respect to the manifolds M and $\mathbb{R}^k$. If N is a unit sphere, then (4) becomes $\Delta_p f = \lambda f$ for a function $\lambda\colon \Omega \longrightarrow \mathbb{R}$. Multiplying the equation with f one obtains $\lambda = -|\nabla f|^p$.

For $p > 2$ the p-Laplace operator is degenerated elliptic. (Weak) solutions of (3) are called (weakly) p-harmonic maps.

On the other hand, vanishing variations of the form

$$f_\varepsilon(x) = f\big(x + \varepsilon\varphi(x)\big) \tag{5}$$

with $\varphi \in C_0^\infty\big(B_\rho(x), \mathbb{R}^m\big)$ and $|\varepsilon|$ small enough, lead to

$$0 = \int_\Omega \sqrt{\gamma}(\gamma^{\sigma\rho}\partial_\sigma f^i\partial_\rho f^i)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\partial_\alpha f^j\partial_\beta(\partial_\delta f^j\varphi^\delta)\,dx.$$

If M is locally Euclidean, this can be rewritten as

$$\int_\Omega |\nabla f|^{p-2}\partial_\alpha f^i\partial_\beta f^i\partial_\beta\varphi^\alpha\,dx = \int_\Omega \frac{1}{p}|\nabla f|^p\partial_j\varphi^j\,dx. \tag{6}$$

For a non-constant metric γ this reformulation is in general not possible since

$$h_\delta := \sqrt{\gamma}(\gamma^{\sigma\rho}\partial_\sigma f^i\partial_\rho f^i)^{\frac{p}{2}-1}\gamma^{\alpha\beta}\partial_\alpha f^j\partial_{\beta\delta}f^j$$

is not a gradient then.

Weakly p-harmonic maps in $W^{1,p}(M,N)$ which satisfy (6) are called *stationary p-harmonic* and satisfy the system

$$\partial_\alpha|\nabla f|^p = p\,\partial_\beta(|\nabla f|^{p-2}\partial_\alpha f^j\partial_\beta f^j) \tag{7}$$

in distributional sense. Here, $W^{1,p}(M,N)$ denotes the nonlinear Sobolev space of functions $g \in W^{1,p}(M,\mathbb{R}^k)$ with $g(x) \in N$ for almost every $x \in M$. Notice that (7) is formally obtained from (4) by multiplication with ∇f. So, (smooth) p-harmonic maps are always stationary. This is in general not true for weakly p-harmonic maps and we will see examples for this later on.

The heat flow related to the p-energy is described by

$$\partial_t f - \Delta_p f \perp T_f N \tag{8}$$

$$f|_{t=0} = f_0 \tag{9}$$

or explicitly for (8)

$$\partial_t f - \Delta_p f = \big(pe(f)\big)^{1-\frac{2}{p}} A(f)(\nabla f, \nabla f) \tag{10}$$

where $A(f)(\cdot,\cdot)$ is the second fundamental form on N. For $p = 2$ Eells and Sampson showed in their famous work [6] of 1964, that there exist global solutions of (8)–(9) provided N has non-positive sectional curvature and that the flow tends for suitable $t_k \longrightarrow \infty$ to a harmonic map. Existence and uniqueness of partially regular solutions of the harmonic flow on Riemannian surfaces (*i.e.* $p = m = 2$) has been shown by Struwe in [10] and recently Freire [7] proved uniqueness in this case in the class of weak solutions. Existence for $p = 2$ in higher dimensions (*i.e.* $m > 2$) has been obtained by Chen and Struwe in [4]. Coron [5] constructed maps u_0 such that the 2-flow $u: B^3 \times \mathbb{R}_+ \longrightarrow S^2$ with initial and boundary data u_0 has infinitely many weak solutions. In fact, Coron showed that for suitable weakly 2-harmonic maps $u_0: B^3 \longrightarrow S^2$ the construction of Chen [2] and Chen-Struwe [4] leads to a weak solution $\underline{u}(x,t)$ of the flow which satisfies a certain monotonicity property (see [11] or [4]) in contrast to $\overline{u}(x,t) := u_0(x)$ which is also a weak solution of the flow. Since a monotonicity formula is not available for the p-harmonic flow for $p > 2$, this approach cannot be carried over to the latter situation. Recently, Coron's result has been reproved in [1] by a different technique. In this paper, we prove non-uniqueness of the p-harmonic flow in case $p > 2$ by combining ideas of [1] and [9].

We will establish the following theorem:

THEOREM 1. *If $f_0: \Omega \subset \mathbb{R}^m \longrightarrow S^n$ is a weakly p-harmonic map from a bounded smooth domain Ω of $\mathbb{R}^m$ (with $2 < p < m$) into a sphere, and if f_0 is not stationary p-harmonic then there exist infinitely many weak solutions of the p-harmonic flow with initial and boundary data f_0, i.e., there are infinitely many global weak solutions $f: \Omega \times \mathbb{R}_+ \longrightarrow S^n$ of*

$$\partial_t f - \operatorname{div}\left(|\nabla f|^{p-2} \nabla f\right) = |\nabla f|^p f \quad \textit{weakly on } \Omega \times \mathbb{R}_+ \tag{11}$$

$$f = f_0 \quad \textit{on the parabolic boundary of } \Omega \times \mathbb{R}_+. \tag{12}$$

REMARKS. (a) Ω can be replaced by any smooth compact Riemannian manifold M which is locally flat.

(b) In the last section we will actually construct examples of non-stationary weakly p-harmonic maps.

2. **Existence of a global weak solution.** In this section we establish a special case of the following theorem, which is proved in [9]. The approximate solutions of the p-harmonic flow constructed in the proof will be used in Section 3 to establish the existence of multiple solutions.

THEOREM 2. *For $2 < p <$ dim(M) there exists a global weak solution of the p-harmonic flow between Riemannian manifolds M and N for arbitrary initial data having finite p-energy in the case when the target N is a homogeneous space with a left invariant metric. The solution $f\colon M \times [0,\infty) \longrightarrow N$ satisfies the energy inequality*

$$\frac{1}{2}\int_0^T \int_M |\partial_t f|^2 \, dt \, d\mu + \frac{1}{p}\int_M |df(T)|^p \, d\mu \leq \frac{1}{p}\int_M |df(0)|^p \, d\mu \tag{13}$$

for all $T > 0$.

PROOF. (in the case $M = \Omega$ and $N = S^n$):
For $f_0, g \in W^{1,p}(\Omega, S^n)$ fixed,

$$f \in W^{1,p}_{f_0}(\Omega, S^n) := \{w \in W^{1,p}(\Omega, S^n) : w - f_0 \in W^{1,p}_0(\Omega, S^n)\}$$

and $h > 0$ let

$$E_g(f) := \int_\Omega \Big(\frac{1}{p}|\nabla f|^p + \frac{1}{2h}|f-g|^2\Big)\, dx.$$

By the direct method of the calculus of variations we find a function $w \in W^{1,p}_{f_0}(\Omega, S^n)$ such that

$$E_g(w) = \inf_{f \in W^{1,p}_{f_0}(\Omega,S^n)} E_g(f).$$

The set of arguments for which the infimum of E_g is attained is usually denoted by $\arg\min E_g$. Now we define recursively a family $f_i \in W^{1,p}_{f_0}(\Omega, S^n)$ by

$$f_{i+1} \in \arg\min E_{f_i} \quad \text{for } i = 0, 1, \ldots .$$

Notice that f_i is a weak solution of the Euler-Lagrange equation to energy $E_{f_{i-1}}$, *i.e.*, there holds for every $i = 1, 2, \ldots$

$$\Pi_{T_{f_i}S^n}\Big(\frac{1}{h}(f_i - f_{i-1})\Big) - \operatorname{div}(|\nabla f_i|^{p-2}\nabla f_i) = |\nabla f_i|^p f_i \tag{14}$$

in distributional sense and $f_i = f_0$ on $\partial\Omega$ in the trace sense. In (14) $\Pi_{T_f S^n}$ denotes the orthogonal projection onto the tangent space $T_f S^n$.

Since f_i minimizes $E_{f_{i-1}}$ we have in particular $E_{f_{i-1}}(f_i) \leq E_{f_{i-1}}(f_{i-1})$, *i.e.*,

$$\int_\Omega \Big(\frac{1}{p}|\nabla f_i|^p + \frac{1}{2h}\,|f_i - f_{i-1}|^2\Big)\, dx \leq \int_\Omega \frac{1}{p}|\nabla f_{i-1}|^p \, dx. \tag{15}$$

Now we define the function $f^{(h)}\colon \Omega \times \mathbb{R}_+ \longrightarrow S^n$ by

$$f^{(h)}(t,\cdot) := f_i \quad \text{for } t \in \big[ih, (i+1)h\big).$$

Thus, rewriting (14) by using the notation $\partial^{(h)}$ for the forward difference quotient in time with step length h, *i.e.*, $(\partial^{(h)} f)(t,x) = \frac{1}{h}\big(f(t+h,x) - f(t,x)\big)$, we get

$$\Pi_{T_{f^{(h)}}S^n}\partial^{(-h)} f^{(h)} - \operatorname{div}(|\nabla f^{(h)}|^{p-2}\nabla f^{(h)}) = |\nabla f^{(h)}|^p f^{(h)} \tag{16}$$

in distributional sense on $\Omega \times (h, \infty)$ and $f^{(h)} = f_0$ on the parabolic boundary of $\Omega \times \mathbb{R}_+$. Summing up (15) we obtain

$$\frac{1}{2}\int_0^{kh}\int_\Omega |\partial^{(h)} f^{(h)}|^2\,dx\,dt + \frac{1}{p}\int_\Omega |\nabla f^{(h)}(kh)|^p\,dx \le \frac{1}{p}\int_\Omega |\nabla f_0|^p\,dx. \tag{17}$$

So, in particular we see that $\{f^{(h)}\}_{h>0}$ is a bounded set in $L^\infty\big(0, \infty; W^{1,p}(\Omega, S^n)\big)$ and hence every sequence in $\{f^{(h)}\}_{h>0}$ has a subsequence $f_i := f^{(h_i)}$ such that

$$f_i - f_0 \overset{*}{\rightharpoonup} f - f_0 \quad \text{weakly}^* \text{ in } L^\infty\big(0, \infty; W_0^{1,p}(\Omega)\big) \tag{18}$$

for a map $f \in L^\infty(0, \infty; W_{f_0}^{1,p}\big(\Omega, S^n)\big)$.

It is now easy to see, that the difference quotient for *fixed* step length $H > 0$ of the sequence $\{f^{(h)}\}_{H>h>0}$ is bounded in $L^2\big(0, \infty; L^2(\Omega)\big)$ by a constant which does not depend on H. Since the set $\{f^{(h)}\}_{h>0}$ is precompact in the space $L^r\big(0, T; L^r(\Omega)\big)$ for all $r < p^* = \frac{mp}{m-p}$ (see [9, Lemma 2]) we have that $\{\partial^{(H)} f\}_{H>0}$ is bounded in $L^2\big(0, \infty; L^2(\Omega)\big)$ and hence f has a distributional time derivative in the latter space. In fact, as a consequence of the partial integration rule for the discrete operator $\partial^{(h)}$, we obtain

$$\partial^{(h)} f^{(h)} \rightharpoonup \partial_t f \quad \text{weakly in } L^2\big(0, \infty; L^2(\Omega)\big)$$

and moreover

$$\Pi_{T_{f^{(h)}}S^n}\,\partial^{(-h)} f^{(h)} \rightharpoonup \partial_t f \quad \text{weakly in } L^2\big(\varepsilon, \infty; L^2(\Omega)\big) \tag{19}$$

for arbitrary $\varepsilon > 0$. This allows to pass to the limit in the first term in equation (16).

Now, by the compactness result in [8] there exists a sequence $h \longrightarrow 0$ such that

$$\nabla f^{(h)} \longrightarrow \nabla f \quad \text{strongly in } L^q\big(0, T; L^q(\Omega)\big) \text{ for all } q < p \tag{20}$$

and hence (since $\{\nabla f^{(h)}\}_{h>0}$ is bounded in $L^p\big(0, T; L^p(\Omega)\big)$)

$$|\nabla f^{(h)}|^{p-2}\nabla f^{(h)} \rightharpoonup |\nabla f|^{p-2}\nabla f \quad \text{weakly in } L^{p'}\big(0, T; L^{p'}(\Omega)\big). \tag{21}$$

This allows to pass to the limit in the p-Laplace term of equation (16) (and in the boundary condition). Now, we are left with the problem to do this also on the right-hand side of (16). To overcome this difficulty, we use a similar technique as in [3].

By taking the wedge product of (16) with $f^{(h)}$, we get

$$\Pi_{T_{f^{(h)}}S^n}\,\partial^{(-h)} f^{(h)} \wedge f^{(h)} - \operatorname{div}(|\nabla f^{(h)}|^{p-2}\nabla f^{(h)} \wedge f^{(h)}) = 0. \tag{22}$$

Using the previously stated results, we can pass to the limit in (22) and obtain

$$\partial_t f \wedge f - \operatorname{div}(|\nabla f|^{p-2}\nabla f \wedge f) = 0 \tag{23}$$

in distributional sense. A short calculation shows that for every function $\tau \in C_0^\infty(\Omega \times \mathbb{R}_+; \mathbb{R})$ the relation

$$\Big(\partial_t f - \operatorname{div}(|\nabla f|^{p-2}\nabla f) - |\nabla f|^p f\Big) \cdot f\tau = 0 \tag{24}$$

automatically holds in distributional sense, provided $|f| = 1$ a.e. in Ω. Note, that any function $\varphi \in C_0^\infty(\Omega \times \mathbb{R}_+; \mathbb{R}^{n+1})$ can be decomposed in the following way

$$\varphi = f(f \cdot \varphi) - f \wedge (f \wedge \varphi). \tag{25}$$

Using the facts that $f \in L^\infty\big(0,\infty; W^{1,p}(\Omega, S^n)\big)$ and $\partial_t f \in L^2\big(0,\infty; L^2(\Omega)\big)$ and an approximation argument we obtain that $\psi = f \wedge \varphi$ and $\tau = f \cdot \varphi$ are admissible test-functions in (23) and in (24). Subtracting the resulting equations and using (25), we get

$$\partial_t f - \operatorname{div}(|\nabla f|^{p-2}\nabla f) = |\nabla f|^p f$$

in distributional sense. Thus f is a weak solution of (11) and (12).

By passing to the limit in (17) we get the energy inequality stated in the theorem.

3. **Proof of Theorem 1.** We assume that $f_0\colon \Omega \longrightarrow S^n$ is weakly p-harmonic but not stationary. That means that for some index α

$$\partial_\alpha |\nabla f_0|^p \not\equiv p\, \partial_\beta(|\nabla f_0|^{p-2}\partial_\beta f_0^j \partial_\alpha f_0^j). \tag{26}$$

In order to prove Theorem 1, it is sufficient to show that the weak solution f constructed in the previous section is not constant in time. To do this we use a similar idea as in [1].

Using variations of the form (5) for the energy E_g we find that for the approximating solutions $f^{(h)}$ introduced in the previous section there holds

$$\partial^{(-h)} f^{(h)} \cdot \partial_\alpha f^{(h)} + \frac{1}{p}\partial_\alpha |\nabla f^{(h)}|^p = \partial_\beta(|\nabla f^{(h)}|^{p-2}\partial_\alpha f^{(h)} \cdot \partial_\beta f^{(h)}). \tag{27}$$

Let us assume by contradiction that the limit function f is constant

$$f(\cdot, t) = f_0 \tag{28}$$

for all times $t \geq 0$ and hence especially $E_p\big(f(\cdot,t)\big) = E_p(f_0)$. By (20) we may assume that $\nabla f^{(h)} \longrightarrow \nabla f = \nabla f_0$ a.e. on $[0,\infty) \times \Omega$. Thus, using Fatou's Lemma and the energy inequality (17), we have that

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla f_0|^p \, dx\, dt &= \int_0^T \int_\Omega \lim_{h\to 0} |\nabla f^{(h)}|^p \, dx\, dt \\
&\leq \liminf_{h \to 0} \int_0^T \int_\Omega |\nabla f^{(h)}|^p \, dx\, dt \\
&\leq \int_0^T \int_\Omega |\nabla f_0|^p \, dx\, dt
\end{aligned}$$

which implies that

$$\nabla f^{(h)} \longrightarrow \nabla f \text{ strongly in } L^p_{\mathrm{loc}}(\Omega \times [0,\infty]).$$

Using the energy inequality once more we get

$$\begin{aligned}
&\frac{1}{p}\int_0^T \int_\Omega |\nabla f_0|^p \, dx\, dt \\
&\quad \le \liminf_{h\to 0}\Big(\frac{1}{2}\int_0^T\int_0^t\int_\Omega |\partial^{(h)} f^{(h)}(\tau)|^2\, dx\, d\tau\, dt + \frac{1}{p}\int_0^T\int_\Omega |\nabla f^{(h)}(t)|^p\, dx\, dt\Big)\\
&\quad \le \frac{1}{p}\int_0^T\int_\Omega |\nabla f_0|^p\, dx\, dt
\end{aligned}$$

and this yields that

$$\partial^{(h)} f^{(h)} \longrightarrow 0 \quad \text{strongly in } L^2(\Omega \times \mathbb{R}_+).$$

This allows to pass to the limit in (27) and we get

$$\partial_\alpha |\nabla f|^p = p\, \partial_\beta(|\nabla f|^{p-2}\partial_\beta f^j \partial_\alpha f^j)$$

in contradiction to (26) which proves that (28) cannot hold true.

Now we have the two distinct solutions $\tilde f(\cdot, t) = f_0$ and the limit f of the approximating functions $f^{(h)}$. Then

$$f_\tau(x,t) = \begin{cases} f_0(x) & \text{for } t \le \tau \\ f(x, t-\tau) & \text{for } t > \tau \end{cases}$$

is an infinite family of solutions of (11) and (12) and the theorem is proved.

REMARKS. (a) Theorem 1 remains true for an arbitrary homogeneous space N with a left invariant metric as target manifold in place of a sphere: The construction of the approximating solutions and the passage to the limit are described in [9], the second part of the proof (*i.e.* Section 3) remains unchanged.

(b) It would be a challenging problem to investigate whether the inverse implication of Theorem 1 holds true: If $f_0\colon \Omega \longrightarrow N$ is a stationary weakly p-harmonic function, then the solution of (11) and (12) is unique.

4. **Examples of non-stationary p-harmonic maps.** We show now that there exist non-stationary weakly $(m-1)$-harmonic maps $f_0\colon B^m \longrightarrow S^{m-1}$. The idea to use the conformal invariance of the p-energy in dimension p is similar to the construction of Coron in [5].

Let us consider the map

$$w\colon S^{m-1} \longrightarrow S^{m-1}, \quad x \longmapsto \pi^{-1} \circ v \circ \pi(x)$$

where $\pi\colon S^{m-1} \longrightarrow \mathbb{R}^{m-1} \cup \{\infty\} =: \overline{\mathbb{R}}^{m-1}$ is the stereographic projection and $v\colon \overline{\mathbb{R}}^{m-1} \longrightarrow \overline{\mathbb{R}}^{m-1}$ is a bijective conformal map, *i.e.*, v belongs to the Möbius group of $\overline{\mathbb{R}}^{m-1}$ which is generated by dilatations $x \longmapsto \lambda x$, isometric maps and inversions $x \longmapsto x/|x|^2$ (and in two dimensions by the complex Möbius transformations). Hence w is a conformal map.

The function $B^m \longrightarrow S^{m-1}$, $x \longmapsto \frac{x}{|x|}$ belongs to $W^{1,p}$ if $p < m$ and is weakly p-harmonic. Because of the conformal invariance of the p-energy in dimension p the map

$$f_0\colon B^m \longrightarrow S^{m-1}, \quad x \longmapsto w\left(\frac{x}{|x|}\right)$$

is also weakly p-harmonic if $p = m - 1$. Now for f_0 there holds

LEMMA 1. *If f_0 is stationary then*

$$\int_{S^{m-1}} |\nabla w|^{m-1} y \, d\sigma(y) = 0. \tag{29}$$

PROOF. First, we observe that for $p = m - 1$

$$\begin{aligned} \int_{B^m} |\nabla f_0|^p \frac{x}{|x|} \, dx &= \int_{S^{m-1}} |\nabla f_0|^p y \, d\sigma(y) \\ &= \int_{S^{m-1}} |\nabla w|^p y \, d\sigma(y). \end{aligned} \tag{30}$$

We now use $|x| - 1$ as a test-function in (7) and obtain that for every index α

$$\begin{aligned} \int_{B^m} |\nabla f_0|^p \frac{x^\alpha}{|x|} \, dx &= \int_{B^m} |\nabla f_0|^p \partial_\alpha |x| \, dx \\ &= p \int_{B^m} |\nabla f_0|^{p-2} \partial_\alpha f_0 \cdot \underbrace{\partial_\beta f_0 \frac{x^\beta}{|x|}}_{=0} \, dx = 0. \end{aligned} \tag{31}$$

The combination of (30) and (31) gives the desired result. ■

It is easy to see that *e.g.* for a dilatation $v\colon x \longmapsto \lambda x$ with $\lambda \neq 1$ the condition (29) is violated.

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ETH-Zentrum
Departement Mathematik
CH-8092 Zürich
Switzerland
e-mail: buhler@math.ethz.ch