Canad. J. Math. Vol. **54** (2), 2002 pp. 239–262

Elementary Symmetric Polynomials in Numbers of Modulus 1

Donald I. Cartwright and Tim Steger

Abstract. We describe the set of numbers $\sigma_k(z_1,\dots,z_{n+1})$, where $z_1,\dots,z_{n+1}$ are complex numbers of modulus 1 for which $z_1z_2\cdots z_{n+1}=1$, and σ_k denotes the k-th elementary symmetric polynomial. Consequently, we give sharp constraints on the coefficients of a complex polynomial all of whose roots are of the same modulus. Another application is the calculation of the spectrum of certain adjacency operators arising naturally on a building of type $\tilde{A}_n$.

1 Introduction, Applications, and Proof Outline

Let n,k be integers, with $n\ge 1$ and $0\le k\le n+1$. Let $\sigma_k(z_1,\dots,z_{n+1})$ denote the k-th elementary symmetric function in the variables $z_1,\dots,z_{n+1}$:

$$\sigma_k(z_1,\dots,z_{n+1})=\sum_{1\le j_1<\cdots<j_k\le n+1} z_{j_1}z_{j_2}\cdots z_{j_k}. \tag{1.1}$$

Let Z_n denote the set of $n+1$-tuples $z=(z_1,\dots,z_{n+1})$ of complex numbers of modulus 1 for which $z_1z_2\cdots z_{n+1}=1$. Our aim is to describe the set $\Sigma_{n,k}$ of all complex numbers of the form (1.1), where $z\in Z_n$. To this end, let $\varphi_{n,k}(\theta)=\sigma_k(a,b,b,\dots,b)$, where $a=e^{-in\theta/(n+1)}$, $b=e^{i\theta/(n+1)}$ and $0\le\theta\le 2\pi(n+1)$. That is,

$$\varphi_{n,k}(\theta)=\binom{n}{k-1}ab^{k-1}+\binom{n}{k}b^k=\frac{1}{n+1}\binom{n+1}{k}e^{ik\theta/(n+1)}(ke^{-i\theta}+n+1-k). \tag{1.2}$$

Our main result is that $\Sigma_{n,k}$ equals the region bounded by the curve $\varphi_{n,k}$. More precisely, let $\mathcal{S}_{n,k}$ denote the set consisting of (the image of) $\varphi_{n,k}$, together with all the bounded components of the complement of this image. We prove the following:

Main Theorem $\Sigma_{n,k}=\mathcal{S}_{n,k}$.

The set $\mathcal{S}_{n,k}$, in the case $n=9$ and $k=3$, is illustrated in Figure 1. The curve $\varphi_{9,3}$ consists of both the solid and the dotted arcs. Let $R_{n,k}$ denote the radius of the largest disk (centred at the origin) contained in $\mathcal{S}_{n,k}$. We also let $M_{n,k}=\binom{n+1}{k}$ and

Received by the editors January 9, 2001; revised July 13, 2001.
AMS subject classification: Primary: 05E05, 33C45, 30C15; secondary: 51E24.

$\rho_{n,k} = M_{n,k}(n+1-2k)/(n+1)$. We have omitted the subscripts in Figure 1.

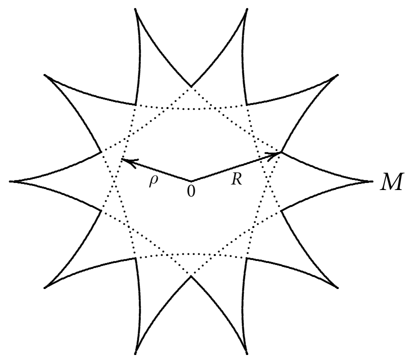

Figure 1

Corollary 1.1 *Let* $x^{n+1} + c_1 x^n + c_2 x^{n-1} + \cdots + c_n x + c_{n+1} \in \mathbb{C}[x]$ *be a monic polynomial whose roots all have the same modulus. Suppose that* $\alpha^{n+1} = c_{n+1}$. *Then* $c_k \in \alpha^k \mathcal{S}_{n,k}$ *for each* k.

Proof If $w_1, \ldots, w_{n+1}$ are the roots of $p(x)$, then $|\alpha| = |w_j|$ for each j. Form $P(X) = p(-\alpha X)/(-\alpha)^{n+1}$. The roots of $P(X)$ are $z_j = -w_j/\alpha$, $j = 1, \ldots, n+1$, all of modulus 1, and the constant term of $P(X)$ is $(-1)^{n+1}$, so that $z_1 z_2 \cdots z_{n+1} = 1$. Hence $P(X)$ equals

$$(1.3)\quad \prod_{j=1}^{n+1}(X - z_j) = X^{n+1} - \sigma_1(z)X^n + \sigma_2(z)X^{n-1} + \cdots + (-1)^n \sigma_n(z)X + (-1)^{n+1}.$$

The coefficient of X^{n+1-k} in $P(X)$ is also $(-1)^k c_k/\alpha^k$. Hence $c_k = \alpha^k \sigma_k(z) \in \alpha^k \Sigma_{n,k}$. ■

The second application is what led us to the present work, but to give a detailed proof would lead us too far from the methods of this paper. We simply give just enough information about a locally finite thick building $\mathfrak{X}$ of type $\tilde{A}_n$ in order to state the result. See [Ca1] for more details. Firstly, $\mathfrak{X}$ is a simplicial complex in which each vertex x has a *type* $\tau(x) \in \{0, 1, \ldots, n\}$. It is natural to define n averaging operators A_k, $k = 1, \ldots, n$, on the Hilbert space of square summable functions f on the vertex set of $\mathfrak{X}$. Here $(A_k f)(x)$ is the average value of $f(y)$ as y runs through the set of neighbours of x of type $\tau(x) + k \bmod n+1$. There is an integer $q \geq 2$ such that the number of such neighbours y is

$$N_{n+1,k} = \frac{(q^{n+1}-1)\cdots(q-1)}{(q^k-1)\cdots(q-1)(q^{n+1-k}-1)\cdots(q-1)}.$$

It has been shown in [Ca1, Ca2] that the A_k's commute, and that they generate a C^*-algebra isomorphic to the space of continuous functions on Z_n symmetric under all permutations of $z_1, \dots, z_{n+1}$. Under this isomorphism, A_k corresponds to the function

$$(z_1, \dots, z_{n+1}) \mapsto \frac{q^{k(n+1-k)/2}}{N_{n+1,k}} \sigma_k(z_1, \dots, z_{n+1}). \tag{1.4}$$

When the building $\mathfrak{X}$ is the one associated with $\mathrm{GL}(n+1, F)$, where F is a local field having a residual field of order q [Ro, Chapter 8] (as must be the case if $n \geq 3$), a proof of this last fact can be given which uses the theory of spherical principal series representations of $\mathrm{GL}(n+1, F)$ and the associated spherical functions (see [Mac1, Mac2]). In any case, it follows that the spectrum of the operator A_k is the image of the function (1.4). Hence

Corollary 1.2 *The spectrum of A_k is the the dilation by $q^{k(n+1-k)/2}/N_{n+1,k}$ of $\mathcal{S}_{n,k}$.*

This result has been proved for the case $n = 2$ (and $k = 1$) in [CM] and [MZ].

Here is an outline of the proof of the main theorem. We start in Section 2 by describing $\mathcal{S}_{n,k}$ in some detail. The main result here is a lower bound for $R_{n,k}$.

If $z = (z_j) \in Z_n$, let N_z denote the number of distinct numbers amongst the z_j's, *i.e.*, $N_z = |\{z_1, \dots, z_{n+1}\}|$. In Section 3, we show that if $z \in Z_n$ and $\sigma_k(z)$ is a boundary point of $\Sigma_{n,k}$, then $N_z \leq 2$. We also show that $\mathcal{S}_{n,k} \subset \Sigma_{n,k}$. In Section 4 we consider $\sigma_k(z)$ when $N_z = 2$. Suppose that $r, s \geq 1$ are integers such that $r+s = n+1$ and let $z = (a^{(r)}, b^{(s)})$ be an $n+1$-tuple consisting of r a's and s b's, where a and b are complex numbers of modulus 1, and $a^r b^s = 1$. We may assume that $r \leq s$. We need to show that $\sigma_k(z)$ is an interior point of $\Sigma_{n,k}$ if $r \geq 2$ and $a \neq b$.

To show, for $a \neq b$ and $r, s \geq 2$, that $\sigma_k(a^{(r)}, b^{(s)})$ is an interior point of $\Sigma_{n,k}$, it is not possible to use standard open mapping theorems, because it turns out that the appropriate Jacobian determinants are zero. So in Section 5 we derive two "*openness conditions*" when $k, r, s \geq 2$:

$$|\sigma_{k-1}(a^{(r-2)}, b^{(s-1)})| > |\sigma_{k-2}(a^{(r-2)}, b^{(s-1)})| \tag{1.5}$$

and

$$|\sigma_{k-1}(a^{(r-1)}, b^{(s-2)})| > |\sigma_{k-2}(a^{(r-1)}, b^{(s-2)})|, \tag{1.6}$$

and show that, if $a \neq b$ and these conditions hold, then $\sigma_k(a^{(r)}, b^{(s)})$ is an interior point of $\Sigma_{n,k}$. Notice that for n, k fixed, (1.5) is just (1.6), with r replaced by $r-1$. The easy case $k = 1$ of the theorem is also dealt with in Section 5.

In Section 6 we finish proving the theorem. The main effort is to show, assuming $k, r \geq 2$, $a \neq b$, and $r < s$, that either the openness conditions hold, or that

$$|\sigma_k(a^{(r)}, b^{(s)})| < R_{n,k}. \tag{1.7}$$

In either case, $\sigma_k(a^{(r)}, b^{(s)})$ is an interior point of $\Sigma_{n,k}$.

There are four cases for which the method described in the previous paragraph does not work: $(n, k, r) = (4, 2, 2)$, $(6, 3, 2)$, $(6, 3, 3)$ and $(8, 4, 2)$. These are dealt with in Section 7.

An important tool in carrying out this program is the expression of $\sigma_k(a^{(r)}, b^{(s)})$ in terms of Jacobi polynomials $P_m^{(\alpha,\beta)}(z)$ for certain integers m, α, β and for z purely imaginary. To treat the case $r, s \geq k$ and the case $s \geq k \geq r$ at the same time, it is convenient to work with generalized Jacobi polynomials $P_m^{(\alpha,\beta)}(z)$ in which the usual conditions $\alpha, \beta > -1$ are not imposed [Sz, Section 4.22]; we will allow $\alpha \leq -1$.

We also use the following obvious symmetries of $\Sigma_{n,k}$:

$$\text{if } w \in \Sigma_{n,k}, \quad \text{then } e^{2\pi ik/(n+1)} w \in \Sigma_{n,k} \text{ and } \bar{w} \in \Sigma_{n,k}. \tag{1.8}$$

For $z \in Z_n$, the complex conjugate of $\sigma_k(z)$ equals $\sigma_{n+1-k}(z)$, and so we can assume that $k \leq (n+1)/2$. The case $n+1 = 2k$ is easy, as then $\Sigma_{n,k}$ is the interval $[-M_{n,k}, M_{n,k}]$, which is also the image of $\varphi_{n,k}$.

We set

$$\varphi_{n,k,r}(\theta) = \sigma_k(a^{(r)}, b^{(s)}) \quad \text{for } a = e^{-is\theta/(n+1)} \text{ and } b = e^{ir\theta/(n+1)},$$

and for $0 \leq \theta \leq 2\pi(n+1)$. This defines a family of (rather beautiful) curves in $\Sigma_{n,k}$. In particular, the curve $\varphi_{n,k}$ above is $\varphi_{n,k,1}$.

Supposing that $a^r b^s = 1$, write $a = e^{-is\theta/(n+1)}$. Then $b = e^{2\pi\nu i/s} e^{ir\theta/(n+1)}$ for some integer ν. Using (4.1) below, $\sigma_k(a^{(r)}, b^{(s)}) = e^{2\pi k\nu i/(n+1)} \sigma_k(c^{(r)}, d^{(s)})$ for $c = e^{-is\theta'/(n+1)}$ and $d = e^{ir\theta'/(n+1)}$, where $\theta' = \theta + 2\pi\nu/s$. So by (1.8), to prove the theorem, we need only show that $\varphi_{n,k,r}(\theta)$ is an interior point of $\Sigma_{n,k}$ when $r \geq 2$ and θ is not a multiple of 2π. Since $\varphi_{n,k,r}(\theta + 2\pi) = e^{2\pi kri/(n+1)} \varphi_{n,k,r}(\theta)$, and since $\varphi_{n,k,r}(-\theta)$ is the complex conjugate of $\varphi_{n,k,r}(\theta)$, by (1.8) again we may assume that $0 < \theta \leq \pi$.

Without loss of generality, we may assume $r \leq s$. Moreover, if $r = s$, then $\varphi_{n,k,r}(\theta) \in \mathbb{R}$, indeed $\varphi_{n,k,r}(\theta) \in [-M_{n,k}, M_{n,k}]$. Given the results of Section 2 below, this makes the case $r = s$ trivial. So in what follows, we can always assume that $r < s$ as well as $k < (n+1)/2$. It turns out that the analysis is most delicate when r is small and k is close to $(n+1)/2$.

2 The Set $\mathcal{S}_{n,k}$

In the next result, we omit the subscripts from $\mathcal{S}_{n,k}$, $R_{n,k}$, $M_{n,k}$, $\varphi_{n,k}$ and $\rho_{n,k}$.

Lemma 2.1 *For all $k < (n+1)/2$, we have $R \geq \max\{\rho, \frac{2}{3\pi} M\}$. If $k > (n+1)/3$, then we also have $R > M\lambda/\sin(3\pi\lambda/2)$ for $\lambda = (n+1-2k)/(n+1)$.*

Proof Consider formula (1.2) for $\varphi(\theta)$. Clearly $|\varphi(\theta)| \leq M$, with equality if and only if $\theta = 2\pi\ell$ for some $\ell \in \{0, 1, \ldots, n+1\}$. Write $\varphi(\theta) = r(\theta) e^{i\psi(\theta)}$, where $r(\theta) \geq 0$, $\psi(\theta) \in \mathbb{R}$ and $\psi(0) = 0$. Since $|ke^{-i\theta} + n + 1 - k| \geq n + 1 - 2k > 0$, it follows that $r(\theta) > 0$, that $r(\theta)$ is differentiable, that we can choose $\psi(\theta)$ differentiable, and that in this case $\psi(2\pi\ell) = 2\pi k\ell/(n+1)$ for $\ell = 0, 1, \ldots, n+1$. Elementary calculations based on the logarithmic derivative show that $\psi(\theta)$ is an increasing function, and that $r(\theta)$ is decreasing on each interval $[2\pi(\ell-1), 2\pi(\ell-1) + \pi]$ and increasing on each interval $[2\pi(\ell-1) + \pi, 2\pi\ell]$. The minimum value of $r(\theta)$ is ρ. Clearly $\mathcal{S}$ contains the disk centred at the origin and having radius ρ. Indeed, this disk is tangent to the curve $\varphi(\theta)$ at the points $\theta = \pi, 3\pi, \dots$.

Let $g = \gcd\{k, n+1\}$. Then the cusps of $\varphi(\theta)$ at $\theta = 2\pi\ell$ and $2\pi\ell'$ coincide if and only if $\ell - \ell'$ is divisible by $(n+1)/g$. So there are exactly $(n+1)/g$ different cusps, at the points $Me^{2\pi gmi/(n+1)}$, $m = 0, \dots, (n+1)/g - 1$. Each of the arcs traced by $\varphi(\theta)$ as θ traverses an interval $[2\pi(\ell-1), 2\pi\ell]$ is traced g times. In particular, if k divides $n+1$, then there is a simple closed curve with $(n+1)/k$ cusps such that $\varphi(\theta)$ traverses this curve k times. For $g < k$, we obtain a simple closed curve from φ as follows. Starting from M, we move along the first arc $\varphi(\theta)$, $0 \le \theta \le 2\pi$, of φ until we meet the ℓ-th arc of φ, where $\ell \in \{1, \dots, (n+1)/g\}$ satisfies $\ell k \equiv g \bmod n+1$. We then follow the ℓ-th arc until it reaches the cusp $Me^{2\pi gi/(n+1)} = \varphi(2\pi\ell)$. Then we start on the $\ell+1$-st arc until it meets the ℓ'-th arc of φ, where $\ell' \in \{1, \dots, (n+1)/g\}$ satisfies $\ell' k \equiv 2g \bmod n+1$. We continue this until we return to M. In Figure 2, we indicate the first and the ℓ-th arcs of φ.

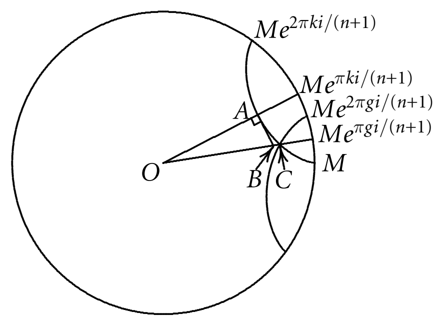

Figure 2

Clearly $R \ge \rho$. If $k \le (n+1)/3$, then $\rho = M\lambda \ge M/3 > M(2/3\pi)$. So suppose that $k > (n+1)/3$. Referring to Figure 2, it is now clear that $R = |\,\mathrm{OC}\,|$, and so $R = |\,\mathrm{OC}\,| > |\,\mathrm{OB}\,| = \rho/\cos\big(\pi(k-g)/(n+1)\big)$. Now $g = \gcd\{k, n+1\} \le \gcd\{2k, n+1\} = \gcd\{n+1-2k, n+1\} \le n+1-2k = (n+1)\lambda$, and so, using $3\pi\lambda/2 \le \pi/2$,

$$\cos\Big(\frac{\pi(k-g)}{n+1}\Big) = \sin\Big(\pi\Big(\frac{\lambda}{2} + \frac{g}{n+1}\Big)\Big) \le \sin\Big(\frac{3\pi\lambda}{2}\Big) \le \frac{3\pi\lambda}{2}.$$

The result follows. ■

3 Reduction to the Case $N_z \le 2$

Let $Z = Z_n$, defined in the introduction. Let S_{n+1} denote the symmetric group on $n+1$ letters, and let Z/S_{n+1} denote the set of orbits in Z under the natural action of S_{n+1}. Let $\psi\colon Z \to Z/S_{n+1}$ denote the natural map, and give Z/S_{n+1} the quotient topology. It is a compact Hausdorff space. We also denote by Z' the set of $(z_1, \dots, z_{n+1}) \in Z$ such that the z_j's are distinct.

Now let X denote the simplex consisting of the $(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ such that $x_j \geq 0$ for each j and such that $\sum_{j=1}^{n+1} x_j = 1$. We also denote by X' the set of $(x_1, \dots, x_{n+1}) \in X$ such that $x_j > 0$ for each j.

Proposition 3.1 *The spaces Z/S_{n+1} and X are homeomorphic, the homeomorphism mapping Z'/S_{n+1} onto X'.*

Proof We first define $f\colon X \to Z$ as follows. Given $x = (x_1, \dots, x_{n+1}) \in X$, let $s_j = x_1 + \cdots + x_{j-1}$ for $j = 1, \dots, n+1$ (so that $s_1 = 0$), and let $\bar{s} = (s_1 + \cdots + s_{n+1})/(n+1)$. Now let

$$f(x) = (e^{2\pi i(s_1 - \bar{s})}, \dots, e^{2\pi i(s_{n+1} - \bar{s})}).$$

Clearly $f(x) \in Z$, and the map f is continuous. Let $F = \psi \circ f$. Then $F\colon X \to Z/S_{n+1}$ is continuous, and we shall now show that it is a bijection. Since X is compact and Z/S_{n+1} is Hausdorff, this will imply that F is a homeomorphism.

Let $z = (z_1, \dots, z_{n+1}) \in Z$. Fix one of the z_j's, say z_1. Then re-order the z_j's so that they are arranged in anti-clockwise order on the unit circle, starting from z_1. We may therefore write $z_{j+1} = z_j e^{2\pi i x_j}$, $j = 1, \dots, n+1$ (the subscripts are understood to be modulo $n+1$ here and below), where $x = (x_1, \dots, x_{n+1}) \in X$. Let s_j, $j = 1, \dots, n+1$, and $\bar{s}$ be as in the definition of $f(x)$, and let $f(x) = (w_1, \dots, w_{n+1})$. Then $z_j = \xi w_j$ for each j, where $\xi = z_1 e^{2\pi i \bar{s}}$. As both z and $f(x)$ are in Z, we must have $\xi^{n+1} = 1$.

Suppose that, from the z_j's arranged in anti-clockwise order on the circle, we start from z_2 instead of z_1. Proceeding as above, this leads to $x' = (x_2, \dots, x_n, x_{n+1}, x_1) \in X$. Let s'_j and $\bar{s}'$ be defined from x' according to the definition of $f(x')$. Then it is routine to check that $s'_j = s_{j+1} - x_1$ for $j = 1, \dots, n$, and that $s'_{n+1} = 1 - x_1$. It follows that $\bar{s}' = \bar{s} - x_1 + 1/(n+1)$, and so $s'_j - \bar{s}' = s_{j+1} - \bar{s} - 1/(n+1)$ for $j = 1, \dots, n$ and $s'_{n+1} - \bar{s}' = 1 - \bar{s} - 1/(n+1)$. Hence $f(x') = (w'_1, \dots, w'_{n+1})$, where $w'_j = e^{-2\pi i/(n+1)} w_{j+1}$ for $j = 1, \dots, n+1$. Repeating this argument, we see that if we start from $z_{\nu+1}$, we get $x^{(\nu)} = (x_{\nu+1}, \dots, x_{n+1}, x_1, \dots, x_\nu) \in X$, for which $f(x^{(\nu)}) = (w_1^{(\nu)}, \dots, w_{n+1}^{(\nu)})$, where $w_j^{(\nu)} = e^{-2\pi \nu i/(n+1)} w_{j+\nu}$ for each j. So if ξ above equals $e^{-2\pi\nu i/(n+1)}$, then $f(x^{(\nu)}) = (z_{\nu+1}, \dots, z_{\nu+n+1})$. Hence $F(x^{(\nu)}) = \psi(z)$, and so F is surjective.

To see that F is injective, suppose that $x = (x_j) \in X$ and $x' = (x'_j) \in X$, and that $F(x') = F(x)$. So $f(x') = (w'_j)$ is a permutation of $f(x) = (w_j)$. If $x_j = 1$ for $j = \ell$ and $x_j = 0$ for all other j, then $w_j = e^{2\pi\ell i/(n+1)}$ for all j, and it is easy to check that x' must equal x. So suppose that x and x' are not of this form. Pick k so that $x'_k > 0$. Then $w'_{k+1} \neq w'_k$. Let $\nu \in \{0, \dots, n\}$ be the largest index such that $w_{k+\nu} = w'_k$. Then as both the w_j's and the w'_j's are distributed in an anticlockwise order around the unit circle, we must have $w'_j = w_{j+\nu}$ for each j. It follows that $x'_j = x_{j+\nu}$ for all j. So $x' = x^{(\nu)}$, and the above calculation shows that $w'_j = e^{-2\pi\nu i/(n+1)} w_{j+\nu}$. Hence $\nu = 0$, so that $x' = x$. So F is injective. The statement about Z'/S_{n+1} and X' is now clear. ∎

Corollary 3.2 $\mathcal{S}_{n,k} \subset \Sigma_{n,k}$

Proof We first show that if $w \in \mathcal{S}_{n,k}$ does not lie on $\varphi_{n,k}$, then the winding number of w with respect to $\varphi_{n,k}$ is positive. For by the first paragraph in the proof of Lemma 2.1, the winding number of 0 is k. If w is on a ray $\{t\zeta : t \geq 0\}$, where $|\zeta| = 1$, which does not pass through the intersection of any two arcs of $\varphi_{n,k}$, then this ray meets $\varphi_{n,k}$ at exactly k distinct points, corresponding to k distinct arcs of $\varphi_{n,k}$. Each time the ray crosses an arc, the winding number is reduced by exactly 1, and so the winding number of w is positive for $w \in \mathcal{S}_{n,k}$. By continuity, the $w \in \mathcal{S}_{n,k}$ not on $\varphi_{n,k}$ which lie on the finitely many remaining rays also have positive winding number.

Suppose that $w \in \mathcal{S}_{n,k} \setminus \Sigma_{n,k}$. The curve $\varphi_{n,k}$ is $\sigma_k \circ \varphi_n$, where $\varphi_n \colon [0, 2\pi(n+1)] \to Z/S_{n+1}$ is the map $\theta \mapsto \psi\big((e^{-in\theta/(n+1)}, e^{i\theta/(n+1)}, \ldots, e^{i\theta/(n+1)})\big)$. By Proposition 3.1, we can contract φ_n to a point in Z/S_{n+1}, and so $\varphi_{n,k}$ contracts in $\Sigma_{n,k}$ to a point. So the winding number of w is zero, contradicting the previous paragraph. ■

Corollary 3.3 *Let* $W = \{(w_1, \ldots, w_n) \in \mathbb{C}^n : \overline{w_k} = w_{n+1-k}$ *for each* $k\}$. *The map* $\sigma \colon \psi(z) \mapsto \big(\sigma_1(z), \ldots, \sigma_n(z)\big)$ *is injective* $Z/S_{n+1} \to W$, *and gives a homeomorphism of* Z'/S_{n+1} *onto an open subset of* W. *If* $z \in Z'$ *and* $n+1 \neq 2k$, *then* $\sigma_k(z)$ *is an interior point of* $\Sigma_{n,k} \subset \mathbb{C}$.

Proof As already noted, $\overline{\sigma_k(z)} = \sigma_{n+1-k}(z)$ for all $z \in Z$. Notice that when $n = 2k-1$ is odd, w_k must be real for any $(w_j) \in W$. For either parity of n, W is homeomorphic to $\mathbb{R}^n$. If $z = (z_j) \in Z$, $z' = (z'_j) \in Z$ and $\sigma_k(z) = \sigma_k(z')$ for each k, then by (1.3) the polynomial $\prod_{j=1}^{n+1}(X - z_j)$ has the same coefficients as $\prod_{j=1}^{n+1}(X - z'_j)$, and hence the same roots. So z' is a permutation of z. Hence σ is injective. By the last proposition, Z'/S_{n+1} is homeomorphic to X', which is homeomorphic to an open subset of $\mathbb{R}^n$. Hence by the invariance of domain theorem (see [Ll, Theorem 3.3.2], for example), σ maps open subsets of Z'/S_{n+1} onto open subsets of W. Since the k-th projection $W \to \mathbb{C}$ is an open map unless $n = 2k - 1$, the last statement is clear. ■

Lemma 3.4 *If* $z \in Z$ *and* $\sigma_k(z)$ *is a boundary point of* $\Sigma_{n,k}$, *then* $N_z \leq 2$.

Proof Suppose that z is a counterexample to the statement for which N_z is maximal. By the last part of Corollary 3.3, we have $3 \leq N_z < n + 1$. Let $a, b, c, \ldots$ be the distinct values of the z_j's and $M_a \geq M_b \geq \cdots$ be their respective multiplicities. Then $M_a \geq 2$. We may suppose that, say, $z_1 = a$, $z_2 = b$ and $z_3 = c$ are distinct. Write

$$\begin{aligned}\sigma_k(z) &= z_1z_2z_3\sigma_{k-3}(z_4, \ldots, z_{n+1}) + (z_1z_2 + z_1z_3 + z_2z_3)\sigma_{k-2}(z_4, \ldots, z_{n+1}) \\ &\quad + (z_1 + z_2 + z_3)\sigma_{k-1}(z_4, \ldots, z_{n+1}) + \sigma_k(z_4, \ldots, z_{n+1}) \\ &= z_1z_2z_3A + (z_1z_2 + z_1z_3 + z_2z_3)B + (z_1 + z_2 + z_3)C + D, \quad \text{say,}\end{aligned}$$

where $A = B = 0$ if $k = 1$ and $A = 0$ if $k = 2$. Let $z_1z_2z_3 = e^{3i\alpha}$, and write $z_j = e^{i\alpha}z'_j$ for $j = 1, 2, 3$. Then z'_1, z'_2 and z'_3 are distinct, of modulus 1, and $z'_1z'_2z'_3 = 1$. So if we let $w = \sigma_1(z'_1, z'_2, z'_3)$, then $\sigma_2(z'_1, z'_2, z'_3)$ is the complex conjugate of w, and so

$$\sigma_k(z) = e^{3i\alpha}A + e^{2i\alpha}\bar{w}B + e^{i\alpha}wC + D.$$

By the above corollary, the point w is an interior point of $\Sigma_{2,1}$, and so there is an $\epsilon > 0$ such that for all $r \in [0, \epsilon]$ and all $\theta \in \mathbb{R}$, there are distinct complex numbers

z_1'', z_2'', z_3'' of modulus 1 such that $z_1''z_2''z_3'' = 1$ and $z_1'' + z_2'' + z_3'' = w + re^{i\theta}$. Given such numbers, let $z_j^* = e^{i\alpha} z_j''$ for $j = 1, 2, 3$, and let $z^* = (z_1^*, z_2^*, z_3^*, z_4, \dots, z_{n+1})$. We get

$$\begin{aligned}\sigma_k(z^*) &= e^{3i\alpha}A + e^{2i\alpha}\overline{(z_1'' + z_2'' + z_3'')}B + e^{i\alpha}(z_1'' + z_2'' + z_3'')C + D\\ &= e^{3i\alpha}A + e^{2i\alpha}(\bar{w} + re^{-i\theta})B + e^{i\alpha}(w + re^{i\theta})C + D\\ &= \sigma_k(z) + r(Be^{i(2\alpha-\theta)} + Ce^{i(\alpha+\theta)}).\end{aligned}$$

Write $B = \rho_B e^{i\theta_B}$ and $C = \rho_C e^{i\theta_C}$, where $\rho_B, \rho_C \geq 0$. Then

$$Be^{i(2\alpha-\theta)} + Ce^{i(\alpha+\theta)} = e^{i\beta}(\rho_B e^{-i(\theta+\gamma)} + \rho_C e^{i(\theta+\gamma)}) \tag{3.1}$$

for $\beta = (\theta_B + \theta_C + 3\alpha)/2$ and $\gamma = (\theta_C - \theta_B - \alpha)/2$. Suppose that $\rho_B \neq \rho_C$. Then as θ increases, the expression in (3.1) traverses the rotation by β radians anticlockwise of the ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a = \rho_B + \rho_C$ and $b = |\rho_B - \rho_C|$. So if we let $r \in [0, \epsilon]$ vary, then $r\big(e^{2i\alpha}Be^{-i\theta} + e^{i\alpha}Ce^{i\theta}\big)$ traverses the scaling down by a factor of ϵ of this ellipse and its interior. So $\sigma_k(z)$ is an interior point of $\Sigma_{n,k}$, contrary to assumption. So we must have $\rho_B = \rho_C$. This cannot happen when $k = 1$, as then $B = 0$ and $C = 1$. So $k \geq 2$ below.

Suppose now that $\rho_B = \rho_C = \rho$. Then $\sigma_k(z^*) = \sigma_k(z) + 2r\rho e^{i\beta}\cos(\theta + \gamma)$. So taking $\theta = \pi/2 - \gamma$, we have $\sigma_k(z^*) = \sigma_k(z)$. Hence, varying $r > 0$, we have infinitely many triples $t^* = (z_1^*, z_2^*, z_3^*)$ of distinct numbers of modulus 1 and product $z_1z_2z_3$ for which $z^* = (z_1^*, z_2^*, z_3^*, z_4, \dots, z_{n+1})$ satisfies $\sigma_k(z^*) = \sigma_k(z)$.

Let $S = \{z_4, \dots, z_{n+1}\}$ and let $N = |S|$. Note that $N_z \leq N + 2$ because $z_1 = a \in S$. There are only finitely many t^* for which $z_1^*, z_2^*, z_3^* \in S$. In fact, there are only finitely many t^* for which $|\{z_1^*, z_2^*, z_3^*\} \cap S| = 2$, since $z_1^*z_2^*z_3^*$ is constant. On the other hand, if $\{z_1^*, z_2^*, z_3^*\} \cap S = \varnothing$, then $N_{z^*} = 3 + N > N_z$, contradicting our choice of z. Hence there is a $j_0 \geq 4$ such that $\{z_1^*, z_2^*, z_3^*\} \cap S = \{z_{j_0}\}$ for infinitely many t^*. Since $N_{z^*} = N + 2$ for the corresponding z^*, and since $N_z \leq N + 1$ if $M_b > 1$, we again get a contradiction unless z is a permutation of an $n+1$-tuple $(a^{(\ell)}, w_{\ell+1}, \dots, w_{n+1})$, where $a, w_{\ell+1}, \dots, w_{n+1}$ are distinct and $a^{(\ell)}$ indicates that a occurs ℓ times in z. We have $\ell = M_a \geq 2$, and may assume that our counterexample z has ℓ as small as possible. This implies that $z_{j_0} = a, = z_1^*$, say. Now

$$\begin{aligned}\sigma_k(z) = \sigma_k(z^*) &= z_2^*z_3^*\sigma_{k-2}(a, z_4, \dots, z_{n+1}) + (z_2^* + z_3^*)\sigma_{k-1}(a, z_4, \dots, z_{n+1})\\ &\quad + \sigma_k(a, z_4, \dots, z_{n+1}).\end{aligned}$$

Since $z_2^*z_3^*$ remains constant and $z_2^* + z_3^*$ changes for different such t^*, we must have $\sigma_{k-1}(a, z_4, \dots, z_{n+1}) = 0$.

We can apply the above reasoning to $(z_1, z_2, z_3) = (a, w_{j_1}, w_{j_2})$, for any distinct $j_1, j_2 \in \{\ell + 1, \dots, n + 1\}$, and get $\sigma_{k-1}(a^{(\ell)}, w_{\ell+1}, \dots, \widehat{w_{j_1}}, \dots, \widehat{w_{j_2}}, \dots, w_{n+1}) = 0$, where $\widehat{w_j}$ indicates that the term w_j is omitted. Now taking any $j_3 \in \{\ell+1, \dots, n+1\}$

distinct from j_1 and j_2, we get

$$\begin{aligned}0 &= \sigma_{k-1}(a^{(\ell)}, w_{\ell+1}, \dots, \widehat{w_{j_1}}, \dots, \widehat{w_{j_2}}, \dots, w_{n+1})\\ &= w_{j_3}\sigma_{k-2}(a^{(\ell)}, w_{\ell+1}, \dots, \widehat{w_{j_1}}, \dots, \widehat{w_{j_2}}, \dots, \widehat{w_{j_3}}, \dots, w_{n+1})\\ &\qquad + \sigma_{k-1}(a^{(\ell)}, w_{\ell+1}, \dots, \widehat{w_{j_1}}, \dots, \widehat{w_{j_2}}, \dots, \widehat{w_{j_3}}, \dots, w_{n+1}).\end{aligned}$$

But we may interchange the roles of j_2 and j_3 here, and as $w_{j_2} \neq w_{j_3}$, we have

$$\begin{aligned}&\sigma_{k-2}(a^{(\ell)}, w_{\ell+1}, \dots, \widehat{w_{j_1}}, \dots, \widehat{w_{j_2}}, \dots, \widehat{w_{j_3}}, \dots, w_{n+1})\\ &\qquad = \sigma_{k-1}(a^{(\ell)}, w_{\ell+1}, \dots, \widehat{w_{j_1}}, \dots, \widehat{w_{j_2}}, \dots, \widehat{w_{j_3}}, \dots, w_{n+1}) = 0.\end{aligned}$$

Starting from the second of these equations and repeating this process, we eventually find that $\sigma_{k-1}(a^{(\ell)}) = 0$, which is impossible, because $\sigma_{k-1}(a^{(\ell)}) = \binom{\ell}{k-1}a^{k-1}$. ■

4 The Case $N_z = 2$

Throughout this section, $r, s \geq 1$ are integers such that $r + s = n + 1$, and a, b are complex numbers of modulus 1, not necessarily satisfying $a^r b^s = 1$. We consider an $n + 1$-tuple $(a^{(r)}, b^{(s)})$ consisting of r a's and s b's. We always assume that $r \leq s$ and $2k < n + 1$, so that $k < s$. It is clear that

$$\sigma_k(a^{(r)}, b^{(s)}) = \sum_{\nu} \binom{r}{\nu}\binom{s}{k-\nu} a^{\nu} b^{k-\nu}, \tag{4.1}$$

where the sum is over the ν for which $0 \leq \nu \leq r$ and $0 \leq k - \nu \leq s$.

We next express $\sigma_k(a^{(r)}, b^{(s)})$ in terms of the Jacobi polynomial $P_m^{(\alpha,\beta)}$, where $(m, \alpha, \beta) = (r, k - r, s - k)$. Notice that if $k < r$ then $\alpha \leq -1$, so that the usual conditions $\alpha, \beta > -1$ placed on Jacobi polynomials are not satisfied. But most of the formulas in [Sz], as noted there, remain valid for arbitrary α and β. The roots of our $P_m^{(\alpha,\beta)}$ are always in $(-1, 1]$. For if $r \leq k$, then the roots are distinct and in $(-1, 1)$ [Sz, Theorem 3.3.1], while if $r > k$ then by [Sz, (4.22.2)], the roots of $P_r^{(k-r,s-k)}$ are those of $P_k^{(r-k,s-k)}$ (all distinct and in $(-1, 1)$), together with $r - k$ 1's.

Lemma 4.1 *Assume that $a \neq b$. Then*

$$\sigma_k(a^{(r)}, b^{(s)}) = \frac{\binom{n+1}{k}}{\binom{n+1}{r}} b^{k-r}(a - b)^r P_r^{(k-r,s-k)}\Big(\frac{a + b}{a - b}\Big). \tag{4.2}$$

Proof This is immediate from (4.1) and [Sz, (4.3.2)], replacing n, α, β and x there by r, $k - r$, $s - k$ and $(a + b)/(a - b)$, respectively. ■

Note that the formula $\sigma_k(a^{(r)}, b^{(s)}) = (a - b)^k P_k^{(r-k,s-k)}\big((a + b)/(a - b)\big)$ can be derived by the same method, but we do not use this.

Corollary 4.2 *Suppose that $a \neq b$, and write $b = ae^{i\theta}$ and $t = \cot(\theta/2)$. Then $|\sigma_k(a^{(r)}, b^{(s)})|$ depends only on t. In fact, let $(m, \alpha, \beta) = (r, k - r, s - k)$, and let*

$x_1, \dots, x_m$ be the zeroes of $P_m^{(\alpha,\beta)}$. Then $\psi_{n,k,r}(t) = |\sigma_k(a^{(r)}, b^{(s)})|$ is given by

$$\psi_{n,k,r}(t)^2 = \binom{n+1}{k}^2 2^{2m} \binom{n+1}{m}^{-2} \frac{|P_m^{(\alpha,\beta)}(it)|^2}{(t^2+1)^m} \tag{4.3}$$

$$= \binom{n+1}{k}^2 \prod_{j=1}^{m} \frac{t^2+x_j^2}{t^2+1}. \tag{4.4}$$

Proof Observe that $it = (1+e^{i\theta})/(1-e^{i\theta})$ and $e^{-i\theta} - 1 = 2/(it-1)$. Then (4.3) is immediate from (4.2). By [Sz, (4.21.6)], the coefficient of z^m in $P_m^{(\alpha,\beta)}(z)$ is $2^{-m}\binom{n+1}{m}$, and since the x_j's are real, we get (4.4) from

$$P_m^{(\alpha,\beta)}(z) = 2^{-m}\binom{n+1}{m}\prod_{j=1}^{m}(z-x_j). \qquad \blacksquare$$

Corollary 4.3 *The function $\psi_{n,k,r}(t)$ is even, and is increasing on $[0,\infty)$.*

Proof Since $x_j \in [-1,1]$ for all j, the derivative of the right hand side of (4.4) is positive. ■

We next need some sums involving the zeroes of $P_m^{(\alpha,\beta)}$.

Lemma 4.4 *Let $x_1, \dots, x_m$ be the zeroes of $P_m^{(\alpha,\beta)}$. Then*

$$\sum_{j=1}^{m} x_j = \frac{m(\beta-\alpha)}{\alpha+\beta+2m}, \tag{4.5}$$

$$\sum_{j=1}^{m}(1-x_j^2) = \frac{4m(\alpha+m)(\beta+m)(\alpha+\beta+m)}{(\alpha+\beta+2m)^2(\alpha+\beta+2m-1)}, \tag{4.6}$$

and

$$\sum_{j=1}^{m}(x_j-x_j^3) = \frac{4m(\alpha+\beta)(\beta-\alpha)(\alpha+m)(\beta+m)(\alpha+\beta+m)}{(\alpha+\beta+2m)^3(\alpha+\beta+2m-1)(\alpha+\beta+2m-2)}. \tag{4.7}$$

Proof Write $P_m^{(\alpha,\beta)}(x) = C(x^m - \sigma_1 x^{m-1} + \sigma_2 x^{m-2} - \sigma_3 x^{m-3} + \cdots)$. By [Sz, (4.21.6)], $C = 2^{-m}\binom{\alpha+\beta+2m}{m}$. One finds using [Sz, (4.21.2)] that σ_1 equals the right hand side of (4.5). Since $\sigma_k = \sigma_k(x_1, \dots, x_m)$ for each k, (4.5) follows. We also get

$$\sigma_2 = \sum_{i<j} x_i x_j = \frac{m(m-1)\big((\alpha-\beta)^2 - (\alpha+\beta) - 2m\big)}{2(\alpha+\beta+2m)(\alpha+\beta+2m-1)},$$

and using $\sum_{j=1}^{m} x_j^2 = \sigma_1^2 - 2\sigma_2$, we get (4.6). Finally, using [Sz, (4.21.2)] again, we get

$$\sigma_3 = \frac{m(m-1)(m-2)(\beta-\alpha)\big((\alpha-\beta)^2 - 3(\alpha+\beta) - 6m + 2\big)}{6(\alpha+\beta+2m)(\alpha+\beta+2m-1)(\alpha+\beta+2m-2)}$$

and then using $\sum_{j=1}^m x_j^3 = 3\sigma_3 + \sigma_1^3 - 3\sigma_1\sigma_2$, we get (4.7). ■

Proposition 4.5 *With notation as in Corollary 4.2, suppose that*

$$\frac{r(n+1-r)}{n(t^2+1)} \geq 2\log\left(\frac{3\pi}{2}\right). \tag{4.8}$$

Then $\psi_{n,k,r}(t) < R_{n,k}$, and so $\sigma_k(a^{(r)}, b^{(s)})$ is an interior point of $\Sigma_{n,k}$.

Proof Write simply $\psi(t)$ for $\psi_{n,k,r}(t)$, and M for $M_{n,k} = \binom{n+1}{k}$. In (4.4), we write $(t^2 + x_j^2)/(t^2+1) = 1 - (1 - x_j^2)/(t^2+1)$, take logarithms, use $-\log(1-u) \geq u$ for any $u < 1$, and get

$$-2\log\left(\frac{\psi(t)}{M}\right) \geq \frac{1}{t^2+1}\sum_{j=1}^m (1 - x_j^2).$$

Applying (4.6) to the (m, α, β) of Corollary 4.2, we get

$$-\log\left(\frac{\psi(t)}{M}\right) \geq \frac{1}{t^2+1}\frac{2kr(n+1-k)(n+1-r)}{n(n+1)^2}. \tag{4.9}$$

Now consider the cases $k/(n+1) \leq 1/3$ and $k/(n+1) > 1/3$ separately. In the first case, we use the fact that $-x(1-x)/\log(1-2x)$ is a decreasing function on the interval $[0, 1/2)$, to see that

$$-\frac{k}{n+1}\left(1 - \frac{k}{n+1}\right) \Big/ \log\left(1 - \frac{2k}{n+1}\right) \geq -\frac{1}{3}\left(1 - \frac{1}{3}\right) \Big/ \log\left(1 - \frac{2}{3}\right) = \frac{2}{9\log(3)}.$$

Hence, using (4.8),

$$\begin{aligned} -\log\left(\frac{\psi(t)}{M}\right) &\geq \frac{1}{t^2+1}\frac{4}{9\log(3)}\frac{r(n+1-r)}{n}\left(-\log\left(1 - \frac{2k}{n+1}\right)\right) \\ &\geq \frac{8\log(3\pi/2)}{9\log(3)}\left(-\log\left(1 - \frac{2k}{n+1}\right)\right) \\ &> -\log\left(1 - \frac{2k}{n+1}\right), \end{aligned}$$

and so

$$\psi(t) < M\left(1 - \frac{2k}{n+1}\right) = \rho_{n,k} \leq R_{n,k}.$$

Now suppose that $k/(n+1) > 1/3$. Write λ for $(n+1-2k)/(n+1)$. Then by Lemma 2.1, we need only show that $\psi(t) \leq M\lambda/\sin(3\pi\lambda/2)$. By (4.9), this will hold if

$$\frac{1}{t^2+1}\frac{r(n+1-r)}{n} \geq \frac{2\log\big(\sin(3\pi\lambda/2)/\lambda\big)}{1-\lambda^2}, \tag{4.10}$$

noting that $k(n+1-k) = (n+1)^2(1-\lambda^2)/4$. We leave it to the reader to verify that the right hand side of (4.10) is a decreasing function of λ on $(0, 1/3)$ and has limit $2\log(3\pi/2)$ as $\lambda \to 0$. So (4.8) implies that (4.10) holds in the case $k/(n+1) > 1/3$.

■

5 The Openness Conditions

To show, for $a \neq b$ and $r, s \geq 2$, that $\sigma_k(a^{(r)}, b^{(s)})$ is an interior point of $\Sigma_{n,k}$, it is not possible to use standard open mapping theorems, because the appropriate Jacobian determinants are zero. One can see this as follows. Let $z = (z_1, \ldots, z_{n+1}) \in Z_n$, and let $\epsilon_j(t)$, $j = 1, \ldots, n+1$, be real valued functions defined and differentiable in an interval about 0. Assume that $\sum_{j=1}^{n+1} \epsilon_j(t) = 0$ for all t and that $\epsilon_j(0) = 0$ for all j. Let $z(t) = (z_1 e^{i\epsilon_1(t)}, \ldots, z_{n+1} e^{i\epsilon_{n+1}(t)})$ and $\sigma(t) = \sigma_k\big(z(t)\big)$. It is not hard to show that $\sigma'(0) = i \sum_{j=1}^{n+1} \sigma_{k-1}(z_1, \ldots, \widehat{z_j}, \ldots, z_{n+1}) z_j \epsilon_j'(0)$. When $z = (a^{(r)}, b^{(s)})$, this shows that $\sigma'(0) = i\big(a\sigma_{k-1}(a^{(r-1)}, b^{(s)}) - b\sigma_{k-1}(a^{(r)}, b^{(s-1)})\big) A = i\xi A$, say, where $A = \sum_{j=1}^{r} \epsilon_j'(0)$. Thus $\sigma'(0)$ is always a real multiple of $i\xi$.

The openness conditions we derive below can be obtained by considering $\sigma''(0)$, but it is simpler to use the next lemma, which appeals to the convexity of the arcs of $\varphi_{n,k}$ for $n = 2$ and $k = 1$. In fact, a routine calculation shows that

(5.1)

$$\varphi_{n,k}'(\theta)\overline{\varphi_{n,k}''(\theta)} = \frac{n^2}{(n+1)^3} \binom{n-1}{k-1}^2 \Big((n+1)\sin(\theta) + i(n+1-2k)\big(1 - \cos(\theta)\big) \Big),$$

which has positive imaginary part. So any arc of any $\varphi_{n,k}$ is convex, in the sense that it lies to the right of the tangent vector to any point of that arc.

Lemma 5.1 *Let $a = e^{i\alpha}$ and $b = e^{i\beta}$, where $\alpha < \beta < \alpha + 2\pi$. Then for $\rho > 0$ sufficiently small and any $t \in [-\pi, \pi]$, there are numbers z_1, z_2, z_3, z_4 of modulus 1 such that $z_1 z_2 z_3 z_4 = a^2 b^2$ and*

$$z_1 + z_2 + z_3 + z_4 = 2a + 2b + \rho e^{i(t + (\alpha+\beta)/2)},$$

where for $0 \leq t \leq \pi$ we can choose $z_4 = b$, and for $-\pi \leq t \leq 0$ we can choose $z_4 = a$.

Proof For $\gamma = (\alpha + 2\beta)/3$ and $\theta = \beta - \alpha$, we have $0 < \theta < 2\pi$, $a = e^{i\gamma} e^{-2i\theta/3}$, $b = e^{i\gamma} e^{i\theta/3}$ and $a + 2b = e^{i\gamma}\varphi_{2,1}(\theta)$. Now $\varphi_{2,1}'(\theta) = (4/3)\sin(\theta/2) e^{-i(\pi+\theta/6)}$, and $\sin(\theta/2) > 0$. By the convexity of the first arc of $\varphi_{2,1}$, there is a $\rho_\theta > 0$ such that $\varphi_{2,1}(\theta) + \rho e^{-i(t+\theta/6)} \in \Sigma_{2,1}$ for $0 \leq \rho \leq \rho_\theta$ and $0 \leq t \leq \pi$. See Figure 3, in which we write φ in place of $\varphi_{2,1}$.

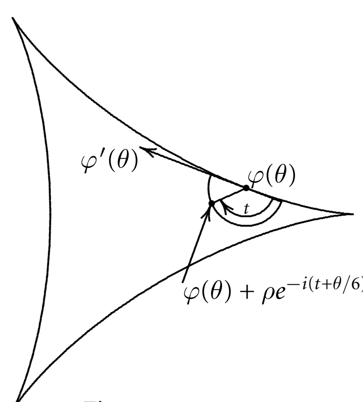

Figure 3

If $(z_1', z_2', z_3') \in Z_2$ and $\sigma_1(z_1', z_2', z_3') = \varphi_{2,1}(\theta) + \rho e^{-i(t+\theta/6)}$, let $z_j = e^{i\gamma} z_j'$ for $j = 1, 2, 3$, and let $z_4 = a$. Then $z_1 z_2 z_3 z_4 = a^2 b^2$ and, using $\gamma - \theta/6 = (\alpha + \beta)/2$, we get

$$z_1 + z_2 + z_3 + z_4 = a + e^{i\gamma}\big(\varphi_{2,1}(\theta) + \rho e^{-i(t+\theta/6)}\big) = 2a + 2b + \rho e^{i(-t+(\alpha+\beta)/2)}.$$

Applying the above to $\bar{b} = e^{-i\beta}$ and $\bar{a} = e^{-i\alpha}$ in place of a and b, respectively, for any $t \in [0, \pi]$ and any small $\rho \geq 0$, we get w_1, w_1, w_3, w_4 such that $w_4 = \bar{b}$, $w_1 w_2 w_3 w_4 = \bar{a}^2 \bar{b}^2$, and $w_1 + w_2 + w_3 + w_4 = 2\bar{a} + 2\bar{b} + \rho e^{i(-t-(\alpha+\beta)/2)}$. So we let $z_j = \bar{w}_j$ for each j. ■

Corollary 5.2 *The main theorem holds in the case $k = 1$.*

Proof Let $z \in Z_n$ and let $\sigma_1(z)$ be a boundary point of $\Sigma_{n,1}$. Then $N_z \leq 2$ by Lemma 3.4. Now suppose that $z = (a^{(r)}, b^{(s)})$, where $a \neq b$ have modulus 1, $s \geq r \geq 1$, $r + s = n + 1$, and $a^r b^s = 1$. Replacing (a, b) by $(\bar{a}, \bar{b})$ if necessary, we may assume that $a = e^{i\alpha}$ and $b = e^{i\beta}$, where $\alpha < \beta < \alpha + 2\pi$. If $r \geq 2$, by replacing two of the a's and two of the b's in z by the z_1, z_2, z_3, z_4 of Lemma 5.1, it is clear that $\sigma_1(a^{(r)}, b^{(s)})$ is an interior point of $\Sigma_{n,1}$. So $r = 1$ must hold. ■

Corollary 5.3 *Suppose that $a \neq b$ have modulus 1, that $k, r, s \geq 2$ and $r + s = n + 1$, and that $a^r b^s = 1$. If (1.5) and (1.6) hold, then $\sigma_k(a^{(r)}, b^{(s)})$ is an interior point of $\Sigma_{n,k}$.*

Proof Writing $a = e^{i\alpha}$ and $b = e^{i\beta}$, we may assume that $\alpha < \beta < \alpha + 2\pi$. For $\rho > 0$ sufficiently small, and for $t \in [-\pi, \pi]$, let z_1, z_2, z_3, z_4 be as in Lemma 5.1. Let $z = z_\rho(t) = (z_1, z_2, z_3, z_4, a^{(r-2)}, b^{(s-2)})$.

Suppose that $0 \leq t \leq \pi$. Write $z' = (a^{(r-2)}, b^{(s-1)})$. Then

$$\sigma_k(z) = z_1 z_2 z_3 \sigma_{k-3}(z') + \sigma_2(z_1, z_2, z_3)\sigma_{k-2}(z') + \sigma_1(z_1, z_2, z_3)\sigma_{k-1}(z') + \sigma_k(z'),$$

omitting the first term on the right if $k = 2$. Using $\sigma_2(z_1, z_2, z_3) = z_1 z_2 z_3 \overline{\sigma_1(z_1, z_2, z_3)}$, $z_1 z_2 z_3 = a^2 b$, and $\sigma_1(z_1, z_2, z_3) = 2a + b + \rho e^{i(t+(\alpha+\beta)/2)}$, and writing $\sigma_{k-1}(z') = \rho_1 e^{i\theta_1}$ and $\sigma_{k-2}(z') = \rho_2 e^{i\theta_2}$, we get

$$\sigma_k(z) = \sigma_k(a^{(r)}, b^{(s)}) + \rho e^{i\delta'}(\rho_1 e^{i\psi'} + \rho_2 e^{-i\psi'})$$

for $\delta' = (\theta_1 + \theta_2 + 2\alpha + \beta)/2$ and $\psi' = t + (\theta_1 - \theta_2 - \alpha)/2$.

Similarly, writing $z'' = (a^{(r-1)}, b^{(s-2)})$, $\sigma_{k-1}(z'') = \rho_3 e^{i\theta_3}$ and $\sigma_{k-2}(z'') = \rho_4 e^{i\theta_4}$, for $-\pi \leq t \leq 0$ we get

$$\sigma_k(z) = \sigma_k(a^{(r)}, b^{(s)}) + \rho e^{i\delta''}(\rho_3 e^{i\psi''} + \rho_4 e^{-i\psi''})$$

where $\delta'' = (\theta_3 + \theta_4 + \alpha + 2\beta)/2$ and $\psi'' = t + (\theta_3 - \theta_4 - \beta)/2$.

The hypotheses (1.5) and (1.6) are that $\rho_1 > \rho_2$ and that $\rho_3 > \rho_4$. Writing $\sigma_k\big(z_\rho(t)\big) = \sigma_k(a^{(r)}, b^{(s)}) + \rho\gamma(t)$, as t increases from $-\pi$ to π, $\gamma(t)$ traverses in the anticlockwise direction a simple closed curve containing 0 in its interior which consists of two half ellipses (which meet at the points $\pm\rho e^{i(\alpha+\beta)/2}\sigma_{k-1}(a^{(r-1)}, b^{(s-1)})$). The result follows. ■

We now express the openness conditions in terms of Jacobi polynomials. Recalling that (1.5) is just (1.6), with r replaced by $r-1$, we shall work with (1.6). In the notation of Corollary 4.2, (1.6) holds if and only if $\psi_{n-3,k-1,r-1}(t) > \psi_{n-3,k-2,r-1}(t)$, which by (4.3) is equivalent to

$$\frac{n-k}{k-1}|P_{r-1}^{(k-r,n-r-k)}(it)| > |P_{r-1}^{(k-r-1,n-r-k+1)}(it)|. \tag{5.2}$$

Lemma 5.4 *Let $a \neq b$, $b = ae^{i\theta}$ and $t = \cot(\theta/2)$. Assume that $t \neq 0$. Then (1.6) holds if and only if*

$$\operatorname{Re}\frac{P'(it)}{P(it)} < 0, \tag{5.3}$$

or, equivalently,

$$-\operatorname{Re}\big(P'(it)P(-it)\big) > 0, \tag{5.4}$$

where $P(z) = P_r^{(k-r-1,n-r-k)}(z)$. When $t = 0$, (1.6) is still equivalent to (5.4), and if $P(0) \neq 0$, to (5.3).

Proof Writing $P_m(z)$ for $P_m^{(k-r-1,n-r-k)}(z)$, and using [Sz, (4.5.4)], we see that (5.2) is equivalent to

$$\frac{n-k}{k-1}|(k-1)P_{r-1}(it) - rP_r(it)| > |(n-k)P_{r-1}(it) + rP_r(it)|. \tag{5.5}$$

As the roots of P_r are real, we have $P_r(it) \neq 0$ if $t \neq 0$. So the last inequality can be written $|w-c| > |w+d|$, for $w = P_{r-1}(it)/P_r(it)$, $c = r/(k-1)$ and $d = r/(n-k)$. For any $w \in \mathbb{C}$, $|w-c| > |w+d|$ if and only if $\operatorname{Re} w < (c-d)/2$. So (5.2) holds if and only if

$$\operatorname{Re}\frac{P_{r-1}(it)}{P_r(it)} < \frac{r(n+1-2k)}{2(k-1)(n-k)}. \tag{5.6}$$

By [Sz, (4.5.7)],

$$(n-1)(t^2+1)P_r'(it) = -r\big((n-1)it + n + 1 - 2k\big)P_r(it) + 2(k-1)(n-k)P_{r-1}(it).$$

Dividing both sides by $P_r(it)$ and taking real parts, we get

$$(n-1)(t^2+1)\operatorname{Re}\frac{P_r'(it)}{P_r(it)} = -r(n+1-2k) + 2(k-1)(n-k)\operatorname{Re}\frac{P_{r-1}(it)}{P_r(it)}.$$

Hence (5.6) is equivalent to (5.3). Clearly (5.3) and (5.4) are equivalent. When $t = 0$, (5.2) implies that $P(0) \neq 0$, because of (5.5), and the argument used above when $t \neq 0$ can be applied. ■

Corollary 5.5 *If $a \neq b$, $b = ae^{i\theta}$ and $t = \cot(\theta/2) \neq 0$, then (1.6) holds if and only if*

$$\sum_{j=1}^{m} \frac{x_j}{t^2 + x_j^2} > 0, \tag{5.7}$$

where $x_1, \ldots, x_m$ are the zeroes of $P_m^{(\alpha,\beta)}(z)$, for $(m, \alpha, \beta) = (r, k - r - 1, n - r - k)$.

Proof This is clear from (5.3) because $P'(z)/P(z) = \sum_{j=1}^{m} 1/(z - x_j)$. ■

Lemma 5.6 *Let $0 \leq \alpha < \beta$, $m \geq 1$, and let $x_1 > \cdots > x_m$ denote the zeroes of $P_m^{(\alpha,\beta)}$. Suppose that f is a real-valued function which is odd and increasing on $[-x_1, x_1]$. Then*

$$\sum_{j=1}^{m} f(x_j) > 0.$$

Proof Let $y_1, \ldots, y_m$ denote the zeroes of the ultraspherical polynomial $P_m^{(\alpha,\alpha)}$, also written in decreasing order. By [Sz, (4.1.3)], if y is a zero of $P_m^{(\alpha,\alpha)}$, then so is $-y$. Since f is odd, the sum $\sum_{j=1}^{m} f(y_j)$ equals 0. But $y_j < x_j$ for all j by [Sz, Theorem 6. 21.1]. So $x_1 > y_1 \geq 0$ and $x_m > y_m = -y_1 > -x_1$, showing that all the x_j's and y_j's lie in $[-x_1, x_1]$. Also, $f(y_j) < f(x_j)$ for each j, since f is an increasing function. The result follows. ■

Corollary 5.7 *For any $0 \leq \alpha < \beta$ and $m \geq 1$, let $x_1 > \cdots > x_m$ denote the zeroes of $P_m^{(\alpha,\beta)}$. Then (5.7) holds if $|t| > x_1$. In particular, (5.7) holds if $|t| \geq 1$. In fact, the sum is positive if $|t| \geq 0.8$.*

Proof Letting $f(x) = x/(t^2+x^2)$, we have $f'(x) = (t^2-x^2)/(t^2+x^2)^2 > 0$ if $|t| > |x|$. So Lemma 5.6 shows that (5.7) holds if $|t| > x_1$, and so if $|t| \geq 1$.

To prove the last statement, let

$$f(x) = \frac{x}{t^2 + x^2} - \frac{1}{3(t^2 + 1)^3}\{(3t^2 - 1)(x - x^3) + (3t^4 + 6t^2 - 5)x\}.$$

Then

$$f'(x) = \frac{(1 - x^2)^2\left(5t^4 + (3t^2 - 1)x^2 + t^2\right)}{(t^2 + 1)^3(t^2 + x^2)^2},$$

which is obviously non-negative for any x if $3t^2 \geq 1$. Hence $\sum_{j=1}^{m} f(x_j) > 0$. But by (4.5) and (4.7), the sums $\sum_{j=1}^{m} x_j$ and $\sum_{j=1}^{m}(x_j - x_j^3)$ are positive. Hence (5.7) holds for any t such that $3t^2 - 1 \geq 0$ and $3t^4 + 6t^2 - 5 \geq 0$. In particular, this is true if $|t| \geq 0.79561$. ■

Corollary 5.8 *Let $a \neq b$, $b = ae^{i\theta}$ and $t = \cot(\theta/2)$. Let $2 \leq k, r < (n + 1)/2$. Then (1.5) holds if $|t| \geq 0.8$, and (1.6) holds if $|t| \geq 0.8$, except perhaps in the case $n = 2r = 2k$, when it holds for $|t| \geq 1$ at least.*

Proof If $r < k$, then applying Corollary 5.7 to $(m, \alpha, \beta) = (r, k-r-1, n-r-k)$, we see that (5.7) and hence (1.6) holds if $|t| \geq 0.8$. If $r \geq k$, then by [Sz, (4.22.2)], the zeroes of $P_r^{(k-r-1,n-r-k)}$ are the zeroes $y_1, \ldots, y_{k-1}$ of $P_{k-1}^{(r+1-k,n-r-k)}$ together with $r+1-k$ 1's. So (5.7) holds if and only if

$$\frac{r+1-k}{t^2+1} + \sum_{j=1}^{k-1} \frac{y_j}{t^2+y_j^2} > 0. \tag{5.8}$$

Assuming that $r+1-k < n-r-k$, *i.e.*, $n > 2r+1$, this is true for $|t| \geq 0.8$ by Corollary 5.7 applied to $(m, \alpha, \beta) = (k-1, r+1-k, n-r-k)$. So (1.6) holds if $|t| \geq 0.8$ if $n > 2r+1$. If $n = 2r+1$, then $r+1-k = n-r-k$, and so the sum $\sum_{j=1}^{k-1} y_j/(t^2+y_j^2)$ is zero (see the proof of Lemma 5.6). So again (5.8) holds if $|t| \geq 0.8$.

When $n = 2r$, then by [Sz, (4.1.3)], the left hand side of (5.8) equals

$$\frac{r+1-k}{t^2+1} - \sum_{j=1}^{k-1} \frac{u_j}{t^2+u_j^2},$$

where $u_1 > \cdots > u_{k-1}$ are the zeroes of $P_{k-1}^{(n-r-k,r+1-k)} = P_{k-1}^{(r-k,r-k+1)}$. If $z_1 > \cdots > z_{k-1}$ are the zeroes of $P_{k-1}^{(r-k,r-k)}$, then by [Sz, Theorem 6.21.1], $z_j < u_j$ for each j. For $f(x) = x/(t^2+x^2)$, $f(u_j) - f(z_j) = f'(\xi_j)(u_j - z_j) \leq (1/t^2)(u_j - z_j)$ for some ξ_j between z_j and u_j. So by (4.5),

$$\begin{aligned} \sum_{j=1}^{k-1} \frac{u_j}{t^2+u_j^2} &= \sum_{j=1}^{k-1} f(u_j) = \sum_{j=1}^{k-1} \big(f(u_j) - f(z_j) \big) \leq \frac{1}{t^2} \sum_{j=1}^{k-1} (u_j - z_j) \\ &= \frac{1}{t^2} \sum_{j=1}^{k-1} u_j = \frac{1}{t^2} \frac{k-1}{2r-1}. \end{aligned}$$

This is less than $(r+1-k)/(t^2+1)$ if $|t| \geq 0.8$ and $k < r$, and if $|t| \geq 1$ and $k = r$.

Replacing r by $r-1$, (1.5) holds if $|t| \geq 0.8$ for any $2 \leq k, r < (n+1)/2$. ■

Lemma 5.9 *The set of $t > 0$ such that (5.4) holds is an interval (t_r, ∞) for some $t_r = t_{n,k,r} \geq 0$. For any n, k (with $2 \leq k < (n+1)/2$), at least one of the numbers t_r and t_{r-1} is 0, and so, given any $b \neq \pm a$, at least one of (1.5) and (1.6) must be true.*

Proof Let $F(t)$ denote the left hand side of (5.4). That is,

$$F(t) = -\frac{1}{2} \big(P'(it)P(-it) + P'(-it)P(it) \big).$$

Then $F'(t) = \big(P''(it)P(-it) - P''(-it)P(it) \big) / 2i$. Let us write

$$\frac{P'(it)}{P(it)} = \sum_{j=1}^{m} \frac{1}{it - x_j} = -\sum_{j=1}^{m} \frac{x_j}{t^2+x_j^2} - it \sum_{j=1}^{m} \frac{1}{t^2+x_j^2} = A(t) + iB(t), \quad \text{say.}$$

Then $P'(it)P(-it) - P'(-it)P(it) = 2i|P(it)|^2B(t)$, and $B(t) < 0$ for all $t > 0$. By [Sz, Theorem 4.2.1],

$$(5.9)\quad (1+t^2)P''(it) + \big((n+1-2k) - (n+1-2r)it\big)P'(it) + r(n-r)P(it) = 0.$$

After some routine manipulations, we get

$$\begin{aligned} 0 &= (1+t^2)\big(P''(it)P(-it) - P''(-it)P(it)\big) + (n+1-2k)\big(P'(it)P(-it) \\ &\quad - P'(-it)P(it)\big) - (n+1-2r)it\big(P'(it)P(-it) + P'(-it)P(it)\big) \\ &= 2i[(1+t^2)F'(t) + (n+1-2k)|P(it)|^2B(t) + (n+1-2r)tF(t)]. \end{aligned}$$

Thus, for any $t > 0$,

$$(5.10)\qquad (1+t^2)F'(t) > -(n+1-2r)tF(t).$$

Suppose that $F(t') > 0$ and that $F(t'') < 0$ for some $t'' > t' \geq 0$. Then if the minimum of $F(t)$ on $[t', t'']$ occurs at $t = t_0$, then $F(t_0) \leq F(t'') < 0$, and so $F'(t_0) > 0$ by (5.10). This is clearly impossible, and so the first statement is proved.

By [Sz, (4.21.7)], $P'(z) = \frac{1}{2}(n-r)P_{r-1}^{(k-r,n+1-r-k)}(z)$, a constant times the polynomial corresponding to $P(z)$, in which r has been replaced by $r-1$. It follows that, if $t \neq 0$, (1.5) holds if and only if

$$\operatorname{Re}\frac{P''(it)}{P'(it)} < 0,$$

Suppose that $t_r > 0$ and $t_{r-1} > 0$. Let $0 < t < t_{r-1}, t_r$. Then $\operatorname{Re} P'(it)/P(it) \geq 0$ and $\operatorname{Re} P''(it)/P'(it) \geq 0$. So $\operatorname{Re} P(it)/P'(it) \geq 0$ too. But this is impossible, because (5.9) implies that

$$(1+t^2)\operatorname{Re}\frac{P''(it)}{P'(it)} + (n+1-2k) + r(n-r)\operatorname{Re}\frac{P(it)}{P'(it)} = 0. \qquad \blacksquare$$

It is clear that the left hand side of (5.4) has the form

$$(5.11)\qquad c_0 + c_1t^2 + \cdots + c_{r-1}t^{2(r-1)}.$$

Although we have not proved it in general, we have observed that for small r, the coefficients $c_1, \ldots, c_{r-1}$ are all positive. When this holds, the first part of Lemma 5.9 is clear. For (5.4) holds for all t if and only if $c_0 > 0$. So $t_r = 0$ when $c_0 > 0$. If $c_0 \leq 0$, t_r is the unique $t \geq 0$ such that the expression in (5.11) is zero. In this case,

$$(5.12)\qquad c_{r-1}t_r^{2(r-1)} \leq c_1t_r^2 + \cdots + c_{r-1}t_r^{2(r-1)} = |c_0|$$

gives a useful estimate for t_r.

6 Proof of the Main Theorem

Recall that our method for proving the main theorem is to show that if a, b are distinct numbers of modulus 1 such that $a^r b^s = 1$, and if $k, r \geq 2$, then either (1.7) or both (1.5) and (1.6) hold. In either case, $\sigma_k(a^{(r)}, b^{(s)})$ is an interior point of $\Sigma_{n,k}$.

We proved the theorem in the case $k = 1$ in Corollary 5.2, and so from now on, we assume that $k \geq 2$.

Proposition 6.1 *Suppose that a, b are distinct numbers of modulus 1 such that $a^r b^s = 1$. Suppose that $2 \leq k, r < (n+1)/2$ and that (n, k, r) is not $(4, 2, 2)$, $(6, 3, 2)$, $(6, 3, 3)$ or $(8, 4, 2)$. Then either (1.7) or both (1.5) and (1.6) hold. Thus $\sigma_k(a^{(r)}, b^{(s)})$ is an interior point of $\Sigma_{n,k}$.*

Proof As usual, write $b = ae^{i\theta}$ and let $t = \cot(\theta/2)$. Suppose first that $r \geq 9$ and $n > 2r$. Then (4.8), and so (1.7), holds for $|t| \leq 0.8$. By Corollary 5.8, both (1.5) and (1.6) hold if $|t| \geq 0.8$.

Next suppose that $r \geq 12$ and $n = 2r$. Then (4.8), and so (1.7), holds for $|t| \leq 1$. By Corollary 5.8, both (1.5) and (1.6) hold if $|t| \geq 1$. The cases $n = 2r$, $r = 9, 10, 11$, are dealt with in Lemma 6.2 below, thereby completing the proof in the case $r \geq 9$.

For $r = 6, 7, 8$, Corollary 5.8 still applies, but (4.8) holds for all $|t| \leq 0.8$ and $n > 2r$ only once $n \geq 33$, $n \geq 22$ and $n \geq 20$, respectively. The cases (i) $r = 6$, $12 \leq n \leq 32$; (ii) $r = 7$, $14 \leq n \leq 21$ and (iii) $r = 8$, $16 \leq n \leq 19$ are dealt with in Lemma 6.2 below, completing the proof of the proposition in the case $r \geq 6$.

We now deal with the cases $r = 2, 3, 4, 5$. The parameter $q = n + 1 - 2k$ is useful here. For $r = 3, 4, 5$, our method involves obtaining estimates for the t_r of Lemma 5.9 of the form $t_r \leq C_r/\sqrt{n}$ for small constants C_r.

The Case $r = 2$ In this case, the openness conditions (1.5) and (1.6) are (5.4) for $r = 1$ and $r = 2$, respectively. When $r = 1$, the left hand side of (5.4) is the positive constant $q(n-1)/4$, and so (1.5) is automatically satisfied. Taking $r = 2$ in (5.4) and dividing by $q(n-2)/32$, we find that condition (1.6) means that

$$(n-1)(n-2)t^2 + q^2 - (n-1) > 0. \tag{6.1}$$

If $q^2 > n - 1$, (6.1) is true for all t, and so both (1.5) and (1.6) hold for all t, and so $\sigma_k(a^{(r)}, b^{(s)})$ is an interior point of $\Sigma_{n,k}$ for all $a \neq b$. Also, $t_2 = 0$ in this case.

If $q^2 \leq n-1$, (6.1) holds if and only if $|t| > t_2$, where t_2 is the positive square root of $\big((n-1) - q^2\big) \big/ \big((n-1)(n-2)\big)$. Write $\psi(t)$ in place of $\psi_{n,k,2}(t)$, in the notation of Corollary 4.2. By Corollary 4.3, to prove the theorem when $r = 2$, it is enough to show that $\psi(t_2) < R_{n,k}$. Taking $r = 2$ in (4.3), and writing $M = \binom{n+1}{k}$ as usual, we find that

$$\frac{q^2}{(n+1)^2} - \frac{\psi(t)^2}{M^2} = \frac{\big((n+1)^2 - q^2\big)\big(q^2 - (nt^2+1)^2\big)}{n^2(n+1)^2(t^2+1)^2},$$

which, when $t = t_2$, equals

$$\frac{\big((n+1)^2 - q^2\big)(n^2q^2 - 4(n-1)^2)}{n^2(n+1)^2\big((n-1)^2 - q^2\big)},$$

which is obviously positive for all $q \geq 2$. Hence $\psi(t_2) < Mq/(n+1) = \rho_{n,k} \leq R_{n,k}$ for $q \geq 2$, by Lemma 2.1. When $q = 1$, we see from (4.3) that

$$\frac{\psi(t_2)^2}{M^2} = \frac{4(n^2+n-1)}{n^2(n+1)^2}.$$

This is less than $(2/3\pi)^2$ for all $n \geq 10$, and so again $\psi(t_2) < R_{n,k}$ in those cases.

Finally, if $4 \leq n \leq 9$, and $q = 1$, the possibilities are $(n,k) = (4,2)$, $(6,3)$ and $(8,4)$. Routine numerical calculations show that $R_{n,k} = 2.5$, 8.15565 and 28.54762, respectively. On the other hand, $\psi(t_2)$ equals 4.359, 10.672 and 29.492. So the cases $(n,k) = (4,2)$, $(6,3)$ and $(8,4)$ need to be treated by different methods (see Section 7 below).

Notice that for all n, k we have shown that $t_2 \leq 1/(n-1)^{1/2}$. For either $q^2 > n-1$, in which case $t_2 = 0$, or $q^2 \leq n-1$, in which case $t_2^2 = (n-1-q^2)/\big((n-1)(n-2)\big) \leq (n-2)/\big((n-1)(n-2)\big) = 1/(n-1)$.

The Case $r = 3$ For $r = 3$, dividing the left hand side of (5.4) by $q(n-3)/768$, we get the condition

$$(6.2) \qquad (n-1-q^2)(3n-5-q^2) + 2(n-2)\big((n-4)q^2 + 2(n-1)\big)t^2 + (n-1)(n-2)^2(n-3)t^4 > 0.$$

The coefficients of t^2 and t^4 being positive, we see that this holds for all t if and only if $q^2 < n-1$ or $q^2 > 3n-5$, in which case $t_3 = 0$. If $n-1 \leq q^2 \leq 3n-5$, then t_3 is the $t \geq 0$ such that the expression on the left in (6.2) equals 0. By (5.12),

$$(n-1)(n-2)^2(n-3)t_3^4 \leq \big(q^2-(n-1)\big)(3n-5-q^2) \leq (n-2)^2,$$

and so $t_3^4 \leq 1/\big((n-1)(n-3)\big) < 1/(n-3)^2$. So $t_3 \leq 1/(n-3)^{1/2}$ for any n, k.

Notice that (4.3) implies that for $(m,\alpha,\beta) = (r, k-r, n+1-k-r)$, and $M = \binom{n+1}{k}$,

$$\psi_{n,k,r}(t) \leq M|p_m^{(\alpha,\beta)}(it)|,$$

where $p_m^{(\alpha,\beta)}$ is the monic polynomial obtained from $P_m^{(\alpha,\beta)}$ by dividing by $2^{-m}\binom{\alpha+\beta+2m}{m}$. Taking $r = 3$, we get $p_3^{(\alpha,\beta)}(x) = x^3 - \sigma_1 x^2 + \sigma_2 x - \sigma_3$, where $\sigma_1 = 3q/(n+1)$, $\sigma_2 = -3(n+1-q^2)big/\big(n(n+1)\big)$ and $\sigma_3 = -q(3n+1-q^2)/\big(n(n^2-1)\big)$.

First assume that $1 \leq q^2 < n-1$. Then $t_3 = 0$ and $t_2 \leq 1/(n-1)^{1/2}$. Using $1 \leq q^2 < n-1$, it is routine to show that $|\sigma_1| \leq 3/(n-1)^{1/2}$, $|\sigma_2| \leq 3/(n-1)$ and that $|\sigma_3| \leq 2/(n-1)^{3/2}$. Hence for $|t| \leq t_2 = \max\{t_2, t_3\}$,

$$|p_3^{(\alpha,\beta)}(it)| \leq 9/(n-1)^{3/2}.$$

This is less than $2/3\pi < R_{n,k}/M$ for all $n \geq 14$. On the other hand, if $|t| > \max\{t_2, t_3\}$, then both (1.5) and (1.6) both hold.

Suppose next that $n-1 \le q^2 \le 3n-5$. Then $\max\{t_2, t_3\} = t_3 < 1/(n-3)^{1/2}$. Using $n-1 \le q^2 \le 3n-5$, we find that $|\sigma_1| \le 3^{3/2}/(n-3)^{1/2}$, $|\sigma_2| \le 6/(n-3)$ and $|\sigma_3| \le 2/(n-3)^{3/2}$. Hence for $|t| \le t_3$,

$$|p_3^{(\alpha,\beta)}(it)| \le (9 + 3^{3/2})/(n-3)^{3/2},$$

which is less than $2/3\pi$ for all $n \ge 20$. If $|t| > t_3$, then both (1.5) and (1.6) hold.

Thus the theorem is proved for $r = 3$ and all $n \ge 20$. The cases $6 \le n \le 19$ are dealt with in Lemma 6.2 below, except for the case $(n, k, r) = (6, 3, 3)$.

The Case $r = 4$ Taking $r = 4$ in (5.4), and dividing by $q(n-4)/36864$, we get the condition $c_0 + c_1 t^2 + c_2 t^4 + c_3 t^6 > 0$, where

$$c_0 = \left(q^2 - (3n-5)\right)\left(q^4 - 2(3n-7)q^2 + 3(n-1)(n-3)\right),$$

$$c_1 = 3(n-3)\left((3n^3 - 22n^2 + 41n - 22) - 2(n^2 - 9n + 10)q^2 + (n-6)q^4\right),$$

$$c_2 = 3(n-4)(n-3)(n-2)\left((n^2 - 1) + (n-5)q^2\right) \quad \text{and}$$

$$c_3 = (n-1)(n-2)^2(n-3)^2(n-4).$$

Since $n \ge 2r = 8$, it is clear that $c_1, c_2, c_3 > 0$. Now write

$$c_0 = \left(q^2 - (3n-5)\right)(q^2 - x_2)(q^2 - x_1)$$

for $x_1, x_2 = 3n - 7 \pm \sqrt{2(3n^2 - 15n + 20)}$. Notice that $x_1 < n - 1 < 3n - 5 < x_2$. So $c_0 > 0$ if $q^2 > x_2$ and if $x_1 < q^2 < 3n - 5$, in which case (1.6) is true for all t, and $t_4 = 0$. If $c_0 \le 0$, t_4 is the solution $t \ge 0$ of $c_0 + c_1 t^2 + c_2 t^4 + c_3 t^6 = 0$. We now estimate t_4 using (5.12).

First, suppose that $3n - 5 \le q^2 \le x_2$. Then $|c_0| = \left(q^2 - (3n-5)\right)(x_2 - q^2) \cdot (q^2 - x_1) \le \left(q^2 - (3n-5)\right)\left((x_2 - x_1)/2\right)^2 \le \left((x_2 - x_1)/2\right)^3 = \left(2(3n^2 - 15n + 20)\right)^{3/2}$. So by (5.12), $t_4^6 \le \left(2(3n^2 - 15n + 20)\right)^{3/2}/c_3 < 6^{3/2}/(n-3)^3$.

Next, suppose that $q^2 \le x_1$. For such q, $|c_0|$ is maximized by taking $q = 1$, and so $|c_0| \le 9(n-2)^2(n-4)$. Thus by (5.12), $t_4^6 \le |c_0|/c_3 \le 9/\left((n-1)(n-3)^2\right) < 6^{3/2}/(n-3)^3$.

Hence we have the estimate $t_4 < 6^{1/4}/(n-3)^{1/2}$ in all cases. Since $t_3 \le 1/(n-3)^{1/2}$, we have $\max\{t_3, t_4\} < 6^{1/4}/(n-3)^{1/2}$ in all cases. So if $|t| > 6^{1/4}/(n-3)^{1/2}$, then both (1.5) and (1.6) hold.

Now (4.8) holds at $t = 6^{1/4}/(n-3)^{1/2}$ once $n \ge 24$. So the theorem is proved when $r = 4$ and $n \ge 24$. The cases $8 \le n \le 23$ and $r = 4$ are dealt with in Lemma 6.2 below.

The Case $r = 5$ Taking $r = 5$ in (5.4) and dividing by $q(n-5)/2949120$, we get the condition $c_0 + c_1 t^2 + c_2 t^4 + c_3 t^6 + c_4 t^8 > 0$ for

$$c_0 = \left(q^4 - 2(3n-7)q^2 + 3(n-1)(n-3)\right)\left(q^4 - 10(n-3)q^2 + 15n^2 - 80n + 89\right),$$

$$c_1 = 4(n-4) \times \big[(n-8)q^6 - 2(3n^2 - 32n + 49)q^4 + (15n^3 - 180n^2 + 529n - 452)q^2 + 6(n-1)(n-3)(5n-9) \big],$$

$$c_2 = 6(n-3)(n-4)\big[(n-6)(n-7)q^4 + 10(n-1)(n-6)q^2 + (n-1)(5n^3 - 50n^2 + 159n - 138)\big],$$

$$c_3 = 4(n-2)(n-3)(n-4)^2(n-5)\big[(n-6)q^2 + 2n(n-1)\big],$$

and

$$c_4 = (n-1)(n-2)^2(n-3)^2(n-4)^2(n-5).$$

It is obvious that $c_2, c_3, c_4 \geq 0$. To see that $c_1 > 0$, note that $c_1 = 4(n-4)f(q^2)$ for a cubic polynomial $f(x)$. The quadratic $f'(x)$ has negative discriminant, and so $f(x)$ is an increasing function. Since $f(0) = 6(n-1)(n-3)(5n-9) > 0$, $f(q^2)$ is positive for all q.

Next notice that c_0 is the product of two quadratics in q^2. If $x_1 < x_2$ are the roots of $x^2 - 2(3n-7)x + 3(n-1)(n-3) = 0$, and $x_3 < x_4$ are the roots of $x^2 - 10(n-3)x + 15n^2 - 80n + 89 = 0$, one finds that $x_1 < x_3 < x_2 < x_4$. So $c_0 > 0$, and therefore $t_5 = 0$, unless $x_2 \leq q^2 \leq x_4$ or $x_1 \leq q^2 \leq x_3$. We now estimate t_5 in these last cases.

If $x_2 \leq q^2 \leq x_4$, then $|c_0| = (q^2-x_1)(q^2-x_2)(q^2-x_3)(x_4-q^2)$. Since $x_3 < q^2 \leq x_4$, the product $(q^2 - x_3)(x_4 - q^2)$ is at most $\big((x_4 - x_3)/2\big)^2 = 2(5n^2 - 35n + 68) \leq 10(n-3)^2$. Also,

$$(x_3 - x_1)(x_3 - x_2) = 4(n-4)(2(n-5) + \sqrt{2(5n^2 - 35n + 68)}) \leq 24(n-3)(n-4).$$

Hence $|c_0| \leq 240(n-3)^3(n-4) < 4^4(n-3)^3(n-4)$. By (5.12), $t_5^8 < 4^4/(n-5)^4$, so that $t_5 < 2/(n-5)^{1/2}$.

Now suppose that $x_1 \leq q^2 \leq x_3$. Then $|c_0| = (q^2 - x_1)(x_2 - q^2)(x_3 - q^2)(x_4 - q^2)$. Now $(q^2 - x_1)(x_2 - q^2) \leq \big((x_2 - x_1)/2\big)^2 = 2(3n^2 - 15n + 20) < 6(n-2)^2$ because q^2 is between x_1 and x_2. Also, $(x_3 - q^2)(x_4 - q^2) \leq (x_3 - x_1)(x_4 - x_1) = 4(n-4)\big(2 + \sqrt{2(3n^2 - 15n + 20)}\big) \leq 12(n-1)(n-4)$. So $|c_0| \leq 72(n-1)(n-2)^2(n-4)$. As $72 < 4^4$, (5.12) again implies that $t_5 < 2/(n-5)^{1/2}$.

Since $t_4 \leq 6^{1/4}/(n-3)^{1/2} < 2/(n-5)^{1/2}$, we have $\max\{t_4, t_5\} < 2/(n-5)^{1/2}$ in all cases. So if $|t| \geq 2/(n-5)^{1/2}$, then both (1.5) and (1.6) hold for $r = 5$.

Now (4.8) holds for $r = 5$ and $t = 2/(n-5)^{1/2}$, provided that $n \geq 20$. This completes the proof of the case $r = 5$ when $n \geq 20$.

The Case $n \leq 32$ The above steps have dealt with the proof of the proposition with the exception of some cases for which $n \leq 32$ and $r \geq 3$. The next statement deals with the remaining cases:

Lemma 6.2 *Suppose that $a \neq b$, $b = ae^{i\theta}$ and $t = \cot(\theta/2)$. If $n \leq 32$, $2 \leq k < (n+1)/2$, $3 \leq r < (n+1)/2$ and $(n,k,r) \neq (6,3,3)$, then at $t = 2/5$, (1.5), (1.6) and (1.7) all hold. Hence for any t either (1.7) or both (1.5) and (1.6) hold.*

Proof We wrote a very simple computer program which verified the first statement. It is based on (4.3) and (5.2) and the estimate for $R_{n,k}$ in Lemma 2.1. We can use the overestimate 22/7 for π, and work entirely with rational numbers. There are 2029 triples (n, k, r) to check (though many of these were eliminated in the above steps, and in fact, fewer than 500 cases remain). The second statement follows from Lemma 5.9 and from the first statement in Corollary 4.3. ■

7 The Last Four Cases

As we have seen, the method of the last section does not apply to the cases $(n, k, r) = (4, 2, 2)$, $(6, 3, 2)$, $(6, 3, 3)$ and $(8, 4, 2)$. In these cases, it is enough to show that $\varphi_{n,k,r}(\theta) \in \mathcal{S}_{n,k}$ for $0 < \theta \leq \pi$ (corresponding to the first half of the first arc of $\varphi_{n,k,r}$).

In Figure 4, the solid curve is the boundary of $\mathcal{S}_{4,2}$, made up of parts of $\varphi_{4,2}$. The dotted curve is $\varphi_{4,2,2}$. We have indicated by an arrow on its first arc the direction in which $\varphi_{4,2,2}$ is traversed. In this case, the openness conditions fail on this arc exactly when $|t| \leq t_2 = 1/\sqrt{3}$, which corresponds to $2\pi/3 \leq \theta \leq 4\pi/3$. In the figure, these points are those between the two bold dots. We need only show that $\varphi_{4,2,2}(\theta) \in \mathcal{S}_{4,2}$ if $2\pi/3 \leq \theta \leq \pi$.

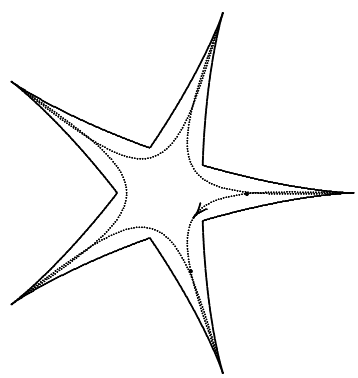

Figure 4

While it is obvious from this figure that (the first half of the first arc of) $\varphi_{4,2,2}$ lies in $\mathcal{S}_{4,2}$, one can give a more formal proof along the following lines. Let $A = \varphi_{4,2}(\pi) = 2e^{\pi i/5}$ and $B = \bar{A}$. Consider the tangents to $\varphi_{4,2}$ at A and B. Let $D = (5/2)e^{\pi i/5}$ be the point where the first and third arcs of $\varphi_{4,2}$ meet, and let $E = \bar{D}$. Consider the quadrilateral Q which is bounded by the above two tangent lines and by the lines through O and D and through O and E. Now $Q \subset \mathcal{S}_{4,2}$ because of the convexity of the arcs of $\varphi_{4,2}$. Using (4.2), one can calculate the principal argument of $\varphi_{4,2,2}(\theta)$, and show that, for $0 \leq \theta \leq \pi$, it is in modulus at most $\pi/5$. This implies that $\varphi_{4,2,2}(\theta)$ lies in the sector bounded by the rays through O and D and through O and E. More direct calculations show that $\varphi_{4,2,2}(\theta)$ lies between the tangents at A and at B provided that $\pi/2 \leq \theta \leq \pi$. Hence $\varphi_{4,2,2}(\theta) \in Q \subset \mathcal{S}_{4,2}$ if $\pi/2 \leq \theta \leq \pi$. We omit further details. See Figure 4b.

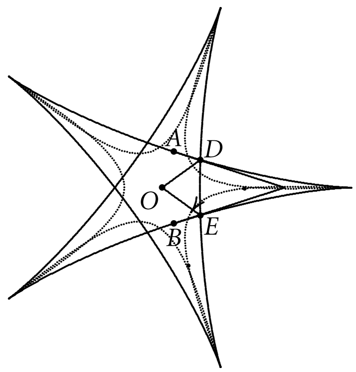

Figure 4b

Similar considerations apply for the other three cases, which we illustrate in the next two figures. In Figure 5, we indicate only one arc of each of $\varphi_{6,3,2}$ and $\varphi_{6,3,3}$.

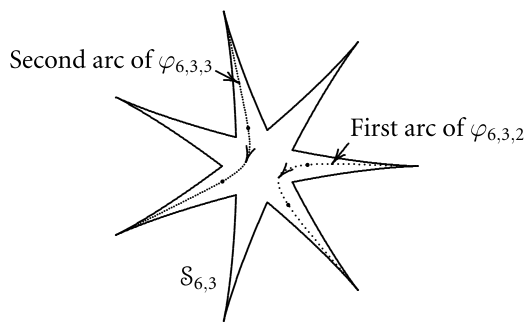

Figure 5

In Figure 6, we indicate $\mathcal{S}_{8,4}$ and all arcs of $\varphi_{8,4,2}$.

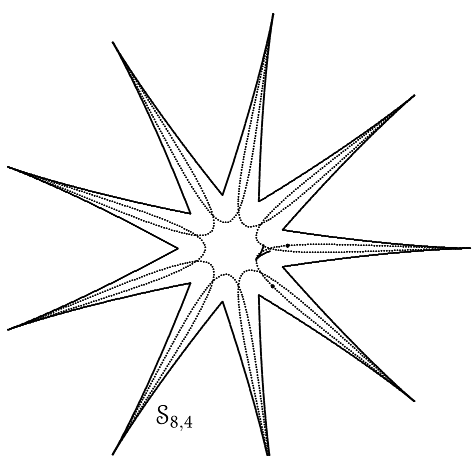

Figure 6

References

[Ca1] D. I. Cartwright, *Harmonic functions on buildings of type $\tilde{A}_n$*. In: Random Walks and Discrete Potential Theory, (eds., M. Picardello and W. Woess), Symposia Math. **XXXIX**, Cambridge Univ. Press, 1999, 104–138.

[Ca2] ______, *Spherical harmonic analysis on buildings of type $\tilde{A}_n$*. Monatsh. Math., to appear.

[CM] D. I. Cartwright and W. Młotkowski, *Harmonic analysis for groups acting on triangle buildings*. J. Austral. Math. **56**(1994), 345–383.

[Ll] N. G. Lloyd, *Degree Theory*. Cambridge Tracts in Math. **73**, Cambridge Univ. Press, 1978.

[Mac1] I. G. Macdonald, *Spherical functions on a group of p-adic type*. Ramanujan Inst. Publications **2** , University of Madras, 1971.

[Mac2] ______, *Symmetric functions and Hall polynomials, second edition*. Oxford Univ. Press, 1995.

[MZ] A. M. Mantero and A. Zappa, *Spherical functions and spectrum of the Laplace operators on buildings of rank* 2. Boll. Un. Mat. Ital. B (6) **8**(1994), 419–475.

[Ro] M. Ronan, *Lectures on buildings*. Academic Press, 1989.

[Sz] G. Szegö, *Orthogonal Polynomials*. Amer. Math. Soc. Colloq. Publ. **XXIII**, Amer. Math. Soc., Providence, Rhode Island, 1939.

School of Mathematics and Statistics
University of Sydney
N.S.W. 2006
Australia

Istituto di Matematica e Fisica
Università di Sassari
via Vienna 2
07100 Sassari
Italy