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Homeomorphic Analytic Maps into the Maximal Ideal Space of H^∞

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Abstract. Let m be a point of the maximal ideal space of H^∞ with nontrivial Gleason part $P(m)$. If $L_m\colon \mathrm{D} \to P(m)$ is the Hoffman map, we show that $H^\infty \circ L_m$ is a closed subalgebra of H^∞. We characterize the points m for which L_m is a homeomorphism in terms of interpolating sequences, and we show that in this case $H^\infty \circ L_m$ coincides with H^∞. Also, if I_m is the ideal of functions in H^∞ that identically vanish on $P(m)$, we estimate the distance of any $f \in H^\infty$ to I_m.

Introduction

In [7] Hoffman characterized the Gleason parts of H^∞ as maximal analytic disks or single points, according to whether the part meets the closure of an interpolating sequence or not. He also showed that among the nontrivial Gleason parts there are those which are homeomorphic to the unit disk and those which are not. Specifically, he proved that if a point $m \in M(H^\infty)$ (the maximal ideal space of H^∞) lies in the closure of a thin interpolating sequence, then the part of m is homeomorphic to the disk. A part satisfying the latter condition is called a homeomorphic disk.

In [5] Gorkin, Lingenberg and Mortini retake the study of homeomorphic disks, obtaining significant new information. In particular, they define a local version of thinness for interpolating sequences that provides a wider class of homeomorphic disks. They also raise several questions, essentially asking whether their condition characterizes homeomorphic disks, and the linked problem of how well behaved (or bad behaved) these parts could be. The first of these problems was recently solved by Izuchi [8], who showed the existence of homeomorphic disks not satisfying the condition in [5]. We will address the other problems. The purpose of this paper is to continue the study of homeomorphic disks. Although in some sense these disks are the least pathological of Gleason parts, we will see that there are different degrees of pathology within the class of homeomorphic disks.

Let us summarize the results in the paper. We look at the functions of H^∞ as functions on its maximal ideal space $M(H^\infty)$. First we show that if $m \in M(H^\infty)$ is a point with nontrivial Gleason part $P(m)$ and $L_m\colon \mathrm{D} \to P(m)$ is the Hoffman map, then $H^\infty \circ L_m$ is a closed subalgebra of H^∞. Then we characterize homeomorphic disks in terms of interpolating sequences satisfying a local condition of weak thinness. We use this characterization to prove that if $P(m)$ is a homeomorphic disk then $H^\infty \circ L_m = H^\infty$. In particular, the map L_m extends to a homeomorphism from $M(H^\infty)$ onto $\overline{P(m)}$. This answers two of the questions in [5] (see also [11]).

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There are two classes of homeomorphic disks, if $P(m)$ satisfies the condition of [5] then the inverse map of L_m is the restriction of a Blaschke product. We will show that in the complementary case any function $f \in H^\infty$ such that $f \circ L_m(z) = z$ for all $z \in \mathbb{D}$ must satisfy $\|f\|_\infty > 1$.

Finally, we show that if $P(m)$ is a homeomorphic disk and $I_m \subset H^\infty$ is the ideal of functions that vanish on $P(m)$, then the distance of any $f \in H^\infty$ to I_m is $\sup\{|f(x)| : x \in P(m)\}$. That is, the best constant in the problem of interpolating bounded analytic functions on $P(m)$ is 1. When $f \circ L_m$ is a non-constant inner function we show another difference between the two types of homeomorphic disks, according to whether the above distance is attained by some $g \in I_m$ or not.

1 Preliminaries

The maximal ideal space $M(H^\infty)$ is identified with the space of nontrivial multiplicative linear functionals on H^∞ with the weak $*$ topology. It is a compact Hausdorff space, and the Gelfand transform, $\hat{f}(x) = x(f)$ for $f \in H^\infty$ and $x \in M(H^\infty)$, establishes an isometry from H^∞ into the algebra of continuous functions on $M(H^\infty)$. We will avoid writing the hat for the Gelfand transform, so looking at H^∞ as an algebra of functions on its maximal ideal space. The pseudohyperbolic metric for $x, y \in M(H^\infty)$ is defined by

$$\rho(x, y) = \sup\{|f(y)| : f \in H^\infty, \|f\|_\infty \leq 1 \text{ and } f(x) = 0\}.$$

Easy consequences of this definition are the Schwarz-Pick inequality, $\rho\big(f(x), f(y)\big) \leq \rho(x, y)$, when $f \in H^\infty$, $\|f\|_\infty \leq 1$, and the formula $\rho(z, \omega) = |z - \omega|/|1 - \overline{\omega} z|$ for $z, \omega \in \mathbb{D}$. Also, it is well known that for any three points z_0, z_1, z_2 in $\mathbb{D}$,

$$\frac{\rho(z_0, z_2) - \rho(z_2, z_1)}{1 - \rho(z_0, z_2)\rho(z_2, z_1)} \leq \rho(z_0, z_1) \leq \frac{\rho(z_0, z_2) + \rho(z_2, z_1)}{1 + \rho(z_0, z_2)\rho(z_2, z_1)}. \tag{1.1}$$

A Blaschke product b with zero sequence $S = \{z_n\}$ is called interpolating if

$$\delta(b) \overset{\text{def}}{=} \inf_k \prod_{n:n \neq k} \rho(z_n, z_k) > 0.$$

Consistently, S is called an interpolating sequence, and $\delta(S) \overset{\text{def}}{=} \delta(b)$. We often use the notation $Z_D(b)$ for the sequence S, and $Z(b) = \{x \in M(H^\infty) : b(x) = 0\}$. It is easy to prove that $Z(b) = \overline{Z_D(b)}$ when b is an interpolating Blaschke product.

The Gleason part of $m \in M(H^\infty)$ is $P(m) = \{x \in M(H^\infty) : \rho(x, m) < 1\}$. Since the condition $\rho(x, m) < 1$ is an equivalence relation, Gleason parts determine a partition of $M(H^\infty)$. In [7] Hoffman defined a continuous map L_m from $\mathbb{D}$ onto $P(m)$, so that $L_m(0) = m$. There are only two possibilities, either $P(m) = \{m\}$ (a trivial part) and L_m is constant, or L_m is one-to-one, which happens if and only if m is in the closure of some interpolating sequence. We write

$$G = \{m \in M(H^\infty) : P(m) \text{ is not a single point}\}.$$

More precisely, $m \in G$ if and only if there is a subnet (z_α) of some interpolating sequence $\{z_n\}$ that tends to m. In this case, the analytic functions $L_{z_\alpha}(z) = (z + z_\alpha)/(1 + z\bar{z}_\alpha)$ tend to $L_m(z)$ in the topology of $M(H^\infty)^{\mathrm{D}}$ (*i.e.*, pointwise). Since any continuous one-to-one function from D onto D is a homeomorphism, the existence of any homeomorphism from D onto $P(m)$ implies that L_m is a homeomorphism. As said in the introduction, a part $P(m)$ for which L_m is a homeomorphism is called a homeomorphic disk.

The map L_m is analytic, in the sense that $f \circ L_m \in H^\infty$ for $f \in H^\infty$. Furthermore, the map $m \mapsto L_m$ from $M(H^\infty)$ into $M(H^\infty)^{\mathrm{D}}$ is continuous, meaning that for any net (m_α) in $M(H^\infty)$ converging to m, $L_{m_\alpha} \to L_m$ pointwise on D.

Let b be a Blaschke product and $x \in Z(b)$. The multiplicity of x as a zero of b is defined as the maximum positive integer n so that $b = b_1 \cdots b_n$, with $b_j(x) = 0$ for $1 \le j \le n$, and it is infinite if there is no such n. The analytical behaviour of $b \circ L_x$ immediately implies that the multiplicity of x is infinite if and only if $b \equiv 0$ on $P(x)$.

2 The Algebra $H^\infty \circ L_m$

For $m \in G \setminus \mathrm{D}$ consider the closed ideal $I_m = \{f \in H^\infty : f \equiv 0 \text{ on } P(m)\}$. The quotient norm makes H^∞/I_m a semisimple Banach algebra with maximal ideal space

$$M(H^\infty/I_m) = \operatorname{hull} I_m = \{x \in M(H^\infty) : f(x) = 0 \text{ for all } f \in I_m\} = \overline{P(m)},$$

(see [4]). Then I_m is the kernel of the map $f \mapsto f \circ L_m$ from H^∞ onto $H^\infty \circ L_m$. Passing to the quotient, we have that the induced map $\Lambda_m : H^\infty/I_m \to \overline{H^\infty \circ L_m}$ is a one-to-one Banach algebras morphism with range $H^\infty \circ L_m$. Additionally, Λ_m is a contraction:

$$\|f \circ L_m\|_\infty = \sup_{x \in P(m)} |f(x)| \le \inf\{\|f + h\|_\infty : h \in I_m\} = \|f + I_m\|_{H^\infty/I_m}.$$

By the open mapping theorem Λ_m is onto (*i.e.*, $H^\infty \circ L_m = \overline{H^\infty \circ L_m}$) if and only if the above inequality has a reciprocal

$$\|f + I_m\|_{H^\infty/I_m} \le K \|f \circ L_m\|_\infty,$$

where K does not depend on f. That is, $H^\infty \circ L_m$ is a closed subalgebra of H^∞ if and only if the above inequality holds. The purpose of this section is to prove this fact. We need an auxiliary lemma, whose proof uses Garnett's reinterpretation of Carleson's proof of the corona theorem. The lemma will be useful in this and the next sections, and it could be of further application. A positive measure μ on D is called a Carleson measure if there is a constant $C > 0$ such that $\mu(Q) \le Cl$ for every sector $Q = \{re^{i\theta} : 1 - l \le r < 1, |\theta - \theta_0| \le l\}$, where $0 < l \le 1$. The infimum of the constants C as above is called the intensity of the measure μ.

Lemma 2.1 *Let u be an inner function and $0 < \beta < 1$. Put $\Omega = \{z \in \mathrm{D} : |u(z)| < \beta\}$ and suppose that $f \in H^\infty(\Omega)$. Then there are $0 < \gamma = \gamma(\beta) < \beta, C = C(\beta) > 0$ and $F \in H^\infty$ such that*

(i) $\|F\|_\infty \le C\|f\|_{H^\infty(\Omega)}$, *and*

(ii) $|F(z) - f(z)| \leq A\|f\|_{H^\infty(\Omega)}|u(z)|$ *when* $|u(z)| < \gamma$, *where* $A = \gamma^{-1}(C+1)$.

Proof By [3, VIII, Thm. 5.1] there are $0 < \gamma = \gamma(\beta) < \beta$ and $\Phi \in C^\infty(\mathrm{D})$ such that

(a) $0 \leq \Phi \leq 1$,
(b) $\Phi(z) = 0$ if $|u(z)| \geq \beta$,
(c) $\Phi(z) = 1$ if $|u(z)| < \gamma$, and
(d) $|\partial\Phi/\partial\overline{z}|\, dx\, dy$ is a Carleson measure with intensity bounded by some constant $K = K(\beta)$.

Henceforth, the Carleson measure $\mu = |(f/u)\partial\Phi/\partial\overline{z}|\, dx\, dy$ has intensity bounded by $\|f\|_{H^\infty(\Omega)}\gamma(\beta)^{-1}K(\beta)$. By [3, VIII, Thm. 1.1] there is $q(z)$ continuous on $\overline{\mathrm{D}}$ and C^∞ on D such that

$$\frac{\partial q}{\partial \overline{z}} = \frac{f}{u}\,\frac{\partial \Phi}{\partial \overline{z}}, \tag{2.1}$$

and

$$\sup_{|z|=1} |q(z)| \leq C_0\gamma^{-1}K\|f\|_{H^\infty(\Omega)}, \tag{2.2}$$

with C_0 an absolute constant. Put $C = C_0\gamma^{-1}K$ and consider the function $F(z) = f(z)\Phi(z) - q(z)u(z)$ $(z \in \mathrm{D})$. Therefore (2.1) implies that $\partial F/\partial\overline{z} = 0$, meaning that F is analytic. Consequently, (b) and (2.2) yield

$$\begin{aligned}\|F\|_\infty &= \lim_{r\to 1^-} \sup\{|F(z)| : |u(z)| \geq r\}\\ &= \lim_{r\to 1^-} \sup\{|q(z)| : |u(z)| \geq r\} \leq C\|f\|_{H^\infty(\Omega)}.\end{aligned}$$

This proves (i). On the other hand, (c) implies that $F(z) = f(z) - q(z)u(z)$ when $|u(z)| < \gamma$. Since u is an open function, by continuity the above equality also holds for $|u(z)| = \gamma$. Thus,

$$\sup_{|u(z)|=\gamma} |q(z)| = \sup_{|u(z)|=\gamma} \frac{|F(z) - f(z)|}{|u(z)|} \leq \gamma^{-1}(C+1)\|f\|_{H^\infty(\Omega)}. \tag{2.3}$$

Moreover, (c) and (2.1) imply that q is analytic on $\{|u| < \gamma\}$. So, by the maximum modulus principle (2.2) and (2.3) yield

$$\sup_{|u(z)|<\gamma} |q(z)| \leq \max\{\sup_{|z|=1} |q(z)|, \sup_{|u(z)|=\gamma} |q(z)|\} \leq A\|f\|_{H^\infty(\Omega)},$$

where $A = \gamma^{-1}(C+1)$. Therefore,

$$|F(z) - f(z)| = |q(z)u(z)| \leq A\|f\|_{H^\infty(\Omega)}\,|u(z)|$$

when $|u(z)| < \gamma$, which proves (ii). ∎

Theorem 2.2 *There is an absolute constant $K \geq 1$ such that for every $m \in G \setminus \mathbb{D}$ and $f \in H^\infty$,*

$$\|f + I_m\|_{H^\infty/I_m} \leq K\|f \circ L_m\|_\infty.$$

In particular, $H^\infty \circ L_m$ is a closed subalgebra of H^∞.

Proof Let $m \in G \setminus \mathbb{D}$ and $f \in H^\infty$ so that $\|f \circ L_m\|_\infty = \sup\{|f(x)| : x \in P(m)\} = 1$. Fix an arbitrary $0 < \beta < 1$ and consider the constant $C(\beta)$ appearing in Lemma 2.1. We will see that there exists $F \in f + I_m$ such that $\|F\|_\infty \leq 2C(\beta) \stackrel{\text{def}}{=} K$.

Since the Shilov boundary $S(H^\infty)$ is disjoint from $\overline{P(m)}$, there is an open neighborhood $U \subset M(H^\infty)$ of $\overline{P(m)}$ such that $\overline{U} \cap S(H^\infty) = \varnothing$ and

$$|f(x)| < 2 \quad \text{for all } x \in U. \tag{2.4}$$

An argument from [6] provides an inner function $u \in I_m$ such that $|u| \geq \beta$ on $M(H^\infty) \setminus U$. Indeed, by [12, Lemma 2.5] there is $h \in I_m$ such that $|h| > 0$ on the compact set $M(H^\infty) \setminus U$. Since h never vanishes on $S(H^\infty)$, its outer factor must be invertible, implying that its inner factor is in I_m. If the singular factor v_s of h is in I_m then a suitable N-root $v_s^{1/N}$ satisfies our requirements. Otherwise there is a Blaschke product $b \in I_m$ (the Blaschke factor of h) so that $|b| > 0$ on $M(H^\infty) \setminus U$. By [13, Thm. 2.5] there is a factorization $b = b_1 \cdots b_n u_1 \cdots u_k$, where each b_j is an interpolating Blaschke product and

$$|u_j| \geq \beta \quad \text{on} \quad M(H^\infty) \setminus U \quad \text{for all } j = 1, \ldots, k.$$

Therefore some of the functions u_j vanishes identically on $\overline{P(m)}$, and this is our function u.

Let $\Omega = \{z \in \mathbb{D} : |u(z)| < \beta\}$. Since $\Omega \subset U \cap \mathbb{D}$ then (2.4) implies that the function $f_1 = f|_\Omega \in H^\infty(\Omega)$ has norm $\|f_1\|_{H^\infty(\Omega)} \leq 2$. Let $F \in H^\infty$ be the function provided by Lemma 2.1 in association to f_1. Part (i) of the lemma yields $\|F\|_\infty \leq C(\beta)\|f_1\|_{H^\infty(\Omega)} \leq C(\beta)2 = K$. By (ii) of the lemma, $|F(z) - f_1(z)| \leq 2A|u(z)|$ when $|u(z)| < \gamma = \gamma(\beta)$. Since the set $\{z \in \mathbb{D} : |u(z)| < \gamma\}$ is contained in Ω (because $\gamma < \beta$) and $f_1 \equiv f$ on Ω, we conclude that

$$|F(z) - f(z)| \leq 2A|u(z)| \quad \text{for } |u(z)| < \gamma.$$

By the corona theorem then $F - f \equiv 0$ on $Z(u)$, and therefore on $\overline{P(m)}$. This means that $F - f \in I_m$, as claimed. ∎

3 Homeomorphic Disks

Let $m \in M(H^\infty) \setminus \mathbb{D}$. The map L_m extends to a continuous map L_m^* from $M(H^\infty)$ onto $\overline{P(m)}$ by the formula $L_m^*(x)(f) = x(f \circ L_m)$, where $x \in M(H^\infty)$ and $f \in H^\infty$ (see [1]). In [5, Thm. 2.1] it is proved that if $P(m)$ is a homeomorphic disk, then $L_m^*|_G$ is a homeomorphism, and the authors ask whether this is also true for L_m^*. A deeper question posed there is whether $H^\infty \circ L_m$ coincides with H^∞ when $P(m)$ is a homeomorphic disk. Clearly, this condition implies the first one. The present section is mainly devoted to give an affirmative answer to this question. From now on we write L_m for either the Hoffman map or its extension L_m^*, the meaning being clear from the context.

In [5, Thm. 1.4] it is shown that $m \in M(H^\infty) \setminus \mathbb{D}$ is such that $P(m)$ is a homeomorphic disk if and only if there is some interpolating sequence S such that $\overline{S} \cap \overline{P(m)} = \{m\}$. The authors also identify a wide class of homeomorphic disks, as we will see next. Let b be a Blaschke product with zeros $\{z_n\}$. We say that b is locally thin at m if there is a subnet (z_α) of $\{z_n\}$ such that $z_\alpha \to m$ and $\lim_\alpha (1 - |z_\alpha|^2)|b'(z_\alpha)| = 1$. A point $m \in M(H^\infty) \setminus \mathbb{D}$ is called locally thin if there exists a Blaschke product b as above. When this happens, by Schwarz lemma $b \circ L_m(z) = \lambda z$ for some constant $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Furthermore, a result of Hoffman [7, Lemma 6.3] now tells us that

$$(f \circ \overline{\lambda} b) \circ L_m = f \quad \text{for every } f \in H^\infty. \tag{3.1}$$

Reciprocally, if there is a Blaschke product b such that $b \circ L_m(z) = z$, then b is locally thin at m, and consequently m is a locally thin point. It is then clear that every locally thin point m lies in a homeomorphic disk (*i.e.*, $P(m)$).

By [7, Thm. 5.3] we can use only interpolating Blaschke products in the definition of locally thin points. If so, by looking at the zero sequence of b we obtain a completely equivalent definition without appealing to nets.

Definition Let $S = \{z_n\}$ be an interpolating sequence and let $m \in \overline{S} \setminus S$. We say that S is locally thin at m if for every $0 < \beta < 1$ there is a subsequence T of S such that $m \in \overline{T}$ and

$$\prod_{n : n \neq k} \rho(z_n, z_k) > \beta \quad \text{for all } z_k \in T.$$

A point $m \in M(H^\infty) \setminus \mathbb{D}$ is locally thin if there is an interpolating sequence as above.

If m is locally thin then so is any other point in $P(m)$. In [5] it is asked if this condition characterizes homeomorphic disks. Although a recent paper of Izuchi [8] shows the existence of homeomorphic disks not containing locally thin points, the characterization can be achieved by slightly relaxing the above definition.

Definition Let S be an interpolating sequence and let $m \in \overline{S} \setminus S$. We say that S is locally weakly thin at m if for every $0 < \beta < 1$ there is a subsequence T of S such that $m \in \overline{T}$ and $\rho(z, S \setminus \{z\}) > \beta$ for every $z \in T$. A point m satisfying this condition is called a locally w-thin point.

Let $m \in G$ and $E \subset \mathbb{D}$ be a subset. We say that m avoids E if for every interpolating sequence S such that $m \in \overline{S}$ and any number $0 < \beta < 1$, there is a subsequence $S_1 \subset S$ such that $m \in \overline{S_1}$ and $\rho(S_1, E) \geq \beta$. Using (1.1) and [7, Thm. 6.1] it is easy to see that if the above condition holds for a fixed interpolating sequence S such that $m \in \overline{S}$ then it holds for every such sequence. This concept was introduced in [14], and its relevance for the present paper is that if $m \in M(H^\infty) \setminus \mathbb{D}$ is in the closure of an interpolating sequence and $y \in M(H^\infty)$, then $y \notin \overline{P(m)}$ if and only if there is some open neighborhood U of y such that m avoids $U \cap \mathbb{D}$ [14, Coro. 2.5].

Lemma 3.1 *Let $m \in M(H^\infty) \setminus \mathbb{D}$ and let R be an interpolating sequence. Then $\overline{R} \cap \overline{P(m)} = \varnothing$ if and only if m avoids R.*

Proof If $\overline{R} \cap \overline{P(m)} = \varnothing$ the above comment says that for every $y \in \overline{R}$ there is an open neighborhood U_y of y such that m avoids $U_y \cap \mathbb{D}$. By compactness there is an open neighborhood U of $\overline{R}$ such that m avoids $U \cap \mathbb{D} \supset R$. Reciprocally, if m avoids R then (3) $\Rightarrow$ (1) in [14, Coro. 2.5] says that $y \notin \overline{P(m)}$ for every $y \in \overline{R}$. ■

***Lemma 3.2* ([7, p. 82])** *Let S be an interpolating sequence, $m \in \overline{S}$, and $0 < \delta < 1$. There exists a subsequence $T \subset S$ such that $m \in \overline{T}$ and $\delta(T) > \delta$.*

Lemma 3.3 *Let S be an interpolating sequence, and let $m \in \overline{S} \setminus S$. Then S is locally w-thin at m if and only if $\overline{S} \cap \overline{P(m)} = \{m\}$.*

Proof Suppose that $\overline{S} \cap \overline{P(m)} = \{m\}$. Lemma 3.1 then implies that m avoids every subsequence $R \subset S$ such that $m \notin \overline{R}$. Let $0 < \beta < 1$. By Lemma 3.2 there is a subsequence T of S such that $m \in \overline{T}$ and $\delta(T) > \beta$. So, m avoids $S \setminus T$, and then there is a subsequence $T_1 \subset T$ such that $m \in \overline{T}_1$ and $\rho(z, S \setminus T) > \beta$ for all $z \in T_1$. Moreover, since $T_1 \subset T$ and $\delta(T) > \beta$, then $\rho(z, T \setminus \{z\}) > \beta$ for all $z \in T_1$. Thus, $\rho(z, S \setminus \{z\}) > \beta$ for all $z \in T_1$.

Now suppose that S is locally w-thin at m, and let $R \subset S$ be an arbitrary subsequence so that $m \notin \overline{R}$. Let $0 < \beta < 1$. By hypothesis there is a subsequence $T \subset S$ such that $m \in \overline{T}$ and $\rho(z, S \setminus \{z\}) > \beta$ for every $z \in T$. We can assume that $T \cap R = \varnothing$ (we take $T \setminus R$ instead of T if necessary). Thus, $\rho(T, R) \geq \beta$, which means that m avoids R. So, no point of $\overline{R}$ is in $\overline{P(m)}$. Hence, whenever $R \subset S$ is a subsequence such that $m \notin \overline{R}$ then $\overline{R} \cap \overline{P(m)} = \varnothing$. Suppose that there is $y \in \overline{S} \cap \overline{P(m)}$, with $y \neq m$, and let $V \subset M(H^\infty)$ be an open neighborhood of y such that $m \notin \overline{V}$. If $R = S \cap V$, we have that $y \in \overline{R} \cap \overline{P(m)}$ and that $m \notin \overline{R}$, a contradiction. ■

Combining [5, Thm. 1.4] and Lemma 3.3 we obtain

Corollary 3.4 *Let $m \in M(H^\infty) \setminus \mathbb{D}$. Then $P(m)$ is a homeomorphic disk if and only if m is a locally w-thin point.*

Before we turn to the main result of this section we need a well known result given by Hoffman (see [7, p. 86 and p. 106], or [3, p. 404]). For $0 < \delta < 1$ consider the two functions

$$\eta(\delta) = \frac{1 - \sqrt{1 - \delta^2}}{\delta} \quad \text{and} \quad \varepsilon(\delta) = \frac{\delta - \eta(\delta)}{1 - \delta\eta(\delta)}\eta(\delta) = \rho\big(\delta, \eta(\delta)\big)\eta(\delta).$$

It is easy to verify that $\eta(\delta)$ and $\varepsilon(\delta)$ take values between 0 and 1, that they increase when δ increases, and that both tend to 1 when $\delta \to 1$. From now on, $\eta(\delta)$ and $\varepsilon(\delta)$ will always mean these functions.

For $z_0 \in \mathbb{D}$ and $1 < r < 1$ put $\Delta(z_0, r) = \{z \in \mathbb{D} : \rho(z, z_0) < r\}$ for the pseudohyperbolic ball with center z_0 and radius r.

Lemma 3.5 *Let b be an interpolating Blaschke product with zero sequence $\{z_n\}$ so that $\delta(b) \geq \delta > 0$. Then*

$$\Delta\big(z_n, \eta(\delta)\big) \cap \Delta\big(z_p, \eta(\delta)\big) = \varnothing \quad \textit{if} \quad n \neq p$$

and

$$|b(z)| \geq \varepsilon(\delta) \quad \textit{for all } z \notin \bigcup_n \Delta\big(z_n, \eta(\delta)\big).$$

Theorem 3.6 *Let $m \in M(H^\infty) \setminus \mathbb{D}$ be a point in a homeomorphic disk and let $f \in H^\infty$. Then there is $F \in H^\infty$ such that*

$$F \circ L_m(z) = f(z) \quad \text{for all } z \in \mathbb{D}.$$

Proof Since m lies in a homeomorphic disk, there is an interpolating sequence S such that $m \in \overline{S}$ and S is locally w-thin at m.

Let $\{\sigma_k\} \subset (0,1)$ be a sequence such that $\beta = \prod_{k\geq 1} \sigma_k > 0$. We choose a sequence $\{\delta_k\} \subset (0,1)$ such that

$$\delta_k < \delta_{k+1} \quad \text{and} \quad \varepsilon(\delta_k) > \sigma_k^{1/k}. \tag{3.2}$$

So, $\delta_k \to 1$ when $k \to \infty$. Once this is done we choose another sequence $\{\alpha_k\} \subset (0,1)$ such that

$$\frac{\eta(\delta_{k-1}) + \eta(\delta_k)}{1 + \eta(\delta_{k-1})\eta(\delta_k)} < \alpha_k. \tag{3.3}$$

By Lemma 3.2 and the definition of locally w-thinness there is a decreasing sequence of interpolating sequences $S \supset S_1 \supset S_2 \supset \cdots$ such that for all $k \geq 1$,

(i) $m \in \overline{S}_k$,
(ii) $\delta(S_k) > \delta_k$,
(iii) $\rho(z, S_1 \setminus \{z\}) > \alpha_k$ for all $z \in S_k$, and
(iv) $\sum_{k\geq 1} \sum_{z\in S_k} k(1 - |z|) < \infty$.

We write $R_k = \{z_n^k : n \geq 1\}$ for the sequence $R_k = S_k \setminus S_{k+1}$.

Claim 1: The pseudohyperbolic balls $\Delta(z_n^k, \eta(\delta_k))$, with $1 \leq n, k$, are pairwise disjoint. If $n \neq p$, then $\Delta(z_n^k, \eta(\delta_k)) \cap \Delta(z_p^k, \eta(\delta_k)) = \varnothing$ by Lemma 3.5, because $z_n^k, z_p^k \in S_k$ and $\delta(S_k) > \delta_k$. If $z_n^k \in R_k$ and $z_p^j \in R_j$ with $j > k$ then $\rho(z_n^k, z_p^j) > \max\{\alpha_k, \alpha_j\}$ by (iii). Suppose that there is $\omega \in \Delta(z_n^k, \eta(\delta_k)) \cap \Delta(z_p^j, \eta(\delta_j))$. Then by (1.1)

$$\rho(z_n^k, z_p^j) \leq \frac{\rho(z_n^k, \omega) + \rho(\omega, z_p^j)}{1 + \rho(z_n^k, \omega)\rho(\omega, z_p^j)} \leq \frac{\eta(\delta_k) + \eta(\delta_j)}{1 + \eta(\delta_k)\eta(\delta_j)}.$$

Since $\{\delta_k\}$ is an increasing sequence, so is $\{\eta(\delta_k)\}$, and since $k \leq j - 1$, the last member of the above inequality is bounded by

$$\frac{\eta(\delta_{j-1}) + \eta(\delta_j)}{1 + \eta(\delta_{j-1})\eta(\delta_j)} < \alpha_j,$$

by (3.3). Thus, $\max\{\alpha_k, \alpha_j\} < \rho(z_n^k, z_m^j) < \alpha_j$, which is a contradiction.

Let b_k be a Blaschke product with zero sequence R_k. Since $R_k \subset S_k$ and $\delta(S_k) > \delta_k$, by Lemma 3.5

$$|b_k(z)| \geq \varepsilon(\delta_k) \quad \text{for all } z \notin \bigcup_{n\geq 1} \Delta(z_n^k, \eta(\delta_k)). \tag{3.4}$$

Let $b = \prod_{k\geq 1} b_k^k$. This Blaschke product converges by condition (iv). In addition, (3.2) and (3.4) tell us that for every $z \notin \Theta \overset{\text{def}}{=} \bigcup_{k\geq 1}\bigcup_{n\geq 1}\Delta\big(z_n^k, \eta(\delta_k)\big)$,

$$|b(z)| = \prod_{k\geq 1}|b_k(z)|^k \geq \prod_{k\geq 1}\varepsilon(\delta_k)^k > \prod_{k\geq 1}\sigma_k = \beta. \tag{3.5}$$

Therefore, $\{z \in \mathrm{D} : |b(z)| < \beta\} \subset \Theta$.

Claim 2: The Blaschke product b ***vanishes on*** $P(m)$***.*** Fix an arbitrary positive integer k_0. Since $S_{k_0} = R_{k_0} \cup R_{k_o+1} \cup \cdots$, and $m \in \overline{S}_{k_0}$, then $\prod_{k\geq k_0} b_k(m) = 0$. Consequently $\prod_{k\geq k_0} b_k^{k_0}$ has a zero of multiplicity k_0 at m, and since this Blaschke product is a factor of b, then m is a zero of b of multiplicity at least k_0. Since k_0 is arbitrary, then m is a zero of infinite multiplicity of b. That is, $b \equiv 0$ on $P(m)$.

We define a bounded analytic function h on the set Θ by the rule

$$h(z) = f \circ L_{z_n^k}^{-1}(z) = f\left(\frac{z - z_n^k}{1 - \overline{z}_n^k z}\right) \quad \text{for } z \in \Delta\big(z_n^k, \eta(\delta_k)\big).$$

Let $0 < r < 1$ be arbitrary. Since $\eta(\delta_k) \to 1$ when $k \to \infty$, then $\eta(\delta_k) > r$ if k is big enough. So, for $\xi \in \Delta(0, r)$ and k that big, we have that $L_{z_n^k}(\xi) \in \Delta\big(z_n^k, \eta(\delta_k)\big)$ and

$$h \circ L_{z_n^k}(\xi) = h\big(L_{z_n^k}(\xi)\big) = f \circ L_{z_n^k}^{-1}\big(L_{z_n^k}(\xi)\big) = f(\xi). \tag{3.6}$$

It is also clear that $\sup_{z\in\Theta}|h(z)| \leq \|f\|_\infty$. We keep these values of r and k. Since $b_k(z_n^k) = 0$ and $L_{z_n^k}$ is isometric with respect to the metric ρ, the Schwarz-Pick inequality yields

$$\begin{aligned} |b_k\big(L_{z_n^k}(\xi)\big)| &= \rho\Big(b_k\big(L_{z_n^k}(\xi)\big), b_k(z_n^k)\Big) \leq \rho\big(L_{z_n^k}(\xi), z_n^k\big) \\ &= \rho\big(L_{z_n^k}(\xi), L_{z_n^k}(0)\big) = \rho(\xi, 0) = |\xi| < r \end{aligned}$$

for all $\xi \in \Delta(0, r)$. Since b_k^k is a factor of b we obtain

$$|b \circ L_{z_n^k}(\xi)| \leq |b_k\big(L_{z_n^k}(\xi)\big)|^k \leq r^k \to 0 \quad \text{when} \quad k \to \infty \tag{3.7}$$

uniformly for $\xi \in \Delta(0, r)$.

Since by (3.5) $\Omega = \{z \in \mathrm{D} : |b(z)| < \beta\} \subset \Theta$, the function $h_1 = h|_\Omega$ is in $H^\infty(\Omega)$ and

$$\|h_1\|_{H^\infty(\Omega)} \leq \|h\|_{H^\infty(\Theta)} \leq \|f\|_\infty.$$

By the above inequality and the fact that $h_1 \equiv h$ on Ω, Lemma 2.1 provides a function $F \in H^\infty$ so that $\|F\| \leq C\|f\|$ and

$$|F(z) - h(z)| \leq A\|f\|\,|b(z)| \quad \text{when} \quad |b(z)| < \gamma, \tag{3.8}$$

where C and $\gamma = \gamma(\beta) < \beta$ are the constants of that lemma.

Claim 3: $F \circ L_m = f$**.** Let $\xi \in \mathbb{D}$ be an arbitrary point and take r so that $|\xi| < r < 1$. By (3.7) there is $k(r)$ such that for every $k \geq k(r)$ we have $\eta(\delta_k) > r$ and $|b(L_{z_n^k}(\xi))| \leq r^k < \gamma$. So, by (3.6), (3.7) and (3.8):

$$\begin{aligned} |F(L_{z_n^k}(\xi)) - f(\xi)| &= |F(L_{z_n^k}(\xi)) - h(L_{z_n^k}(\xi))| \\ &\leq A\|f\|\,|b(L_{z_n^k}(\xi))| \leq C\|f\|r^k. \end{aligned} \tag{3.9}$$

Let $(z_{n(i)}^{k(i)})$ be a subnet of the sequence $\{z_n^k : k, n \geq 1\}$ converging to m. Since $m \in \overline{S}_k$ for every k and disjoint subsequences of an interpolating sequence have disjoint closures, then $m \notin \overline{R}_k$ for any k. Consequently $k(i) \to \infty$.

Therefore $L_{z_{n(i)}^{k(i)}}(z) \to L_m(z)$ for every $z \in \mathbb{D}$, and by (3.9) and the continuity of F on $M(H^\infty)$,

$$F(L_m(z)) = \lim_i F(L_{z_{n(i)}^{k(i)}}(z)) = f(z).$$

This completes the proof of the theorem. ■

4 Norm Estimate for Some Quotients of H^∞

We believe that the best constant K in Theorem 2.2 should be 1, that is, Λ_m should be an isometry. As a partial evidence we show that this holds for homeomorphic disks.

Theorem 4.1 *Let $m \in G \setminus \mathbb{D}$ lying in a homeomorphic disk. Then Λ_m is an isometry. That is, for every $f \in H^\infty$,*

$$\|f + I_m\|_{H^\infty/I_m} = \|f \circ L_m\|_\infty.$$

If $f \circ L_m$ is a non-constant inner function, then there exists $f_1 \in f + I_m$ such that $\|f_1\|_\infty = 1$ if and only if m is a locally thin point.

Observe that if m is a locally thin point then (3.1) implies that the function f_1 in the theorem can be chosen inner. The proof of the theorem requires several auxiliary results. The hyperbolic metric is defined as

$$h(\zeta_1, \zeta_2) = \log\left(\frac{1 + \rho(\zeta_1, \zeta_2)}{1 - \rho(\zeta_1, \zeta_2)}\right) \quad \text{for } \zeta_1, \zeta_2 \in M(H^\infty),$$

(see [3, p. 5]). We use the metric h in the next lemma in order to simplify calculations involving the triangular inequality.

Lemma 4.2 *Let $m \in M(H^\infty) \setminus \mathbb{D}$ be a point in a homeomorphic disk and let $m_0, m_1, \ldots$ be a sequence of different points in $P(m)$ such that $m_0 = m$ and $\xi_k = L_m^{-1}(m_k)$ is a Blaschke sequence in $\mathbb{D}$. Put $\rho_k = \rho(m_k, m)$. Suppose that $\alpha_k, \delta_k \in (0, 1)$ are numbers satisfying*

$$\alpha_{k+1} > \alpha_k > \rho_k \quad \textit{and} \quad \delta_k > 2\rho_k/(1 + \rho_k^2), \tag{4.1}$$

for all $k \geq 0$. Then there are interpolating sequences $T_k = \{\omega_n^k ; n \geq 1\}$ satisfying the following conditions.

(a) *Fix an arbitrary* $\omega_n^k \in T_k$. *Then* $\rho(\omega_n^k, \omega_p^k) > \alpha_k$ *for* $p \neq n$.
If $l < k$ *there exists a unique* $\omega_{n'}^l \in T_l$ *such that* $\rho(\omega_n^k, \omega_{n'}^l) = \rho(m_k, m_l)$, *and* $\rho(\omega_n^k, \omega_p^l) > \alpha_k$ *for* $p \neq n'$.
If $l > k$ *there is at most one* $\omega_{n'}^l \in T_l$ *such that* $\rho(\omega_n^k, \omega_{n'}^l) = \rho(m_k, m_l)$, *and* $\rho(\omega_n^k, \omega_p^l) > \alpha_l$ *for any other* p.

(b) $\delta(T_k) \geq \delta_k$.

(c) $\overline{T}_k \cap \overline{P(m)} = \{m_k\}$.

Proof Since m lies in a homeomorphic disk, by Lemma 3.3 there is an interpolating sequence S such that $m \in \overline{S}$ and S is locally w-thin at m. Combining this fact with Lemma 3.2 we obtain a decreasing chain of sequences $S \supset S_0 \supset S_1 \supset \cdots$ such that

(I) $\overline{S}_k \cap \overline{P(m)} = \{m\}$,

(II) $\delta(S_k) > \beta_k$, where $0 < \beta_k < 1$ is chosen so that $\min\{\beta_k, \rho\big(\beta_k, 2\rho_k/(1+\rho_k^2)\big)\} \geq \delta_k$, and

(III) for every $z \in S_k$, $\rho(z, S \setminus \{z\}) > \gamma_k$, where $0 < \gamma_k < 1$ is given by

$$\frac{1+\gamma_k}{1-\gamma_k} = \left(\frac{1+\alpha_k}{1-\alpha_k}\right) \prod_{1 \leq j \leq k} \left(\frac{1+\rho_j}{1-\rho_j}\right). \tag{4.2}$$

Observe that since $\alpha_{k+1} > \alpha_k$ and $(1+x)/(1-x)$ increases with $x \in (0,1)$, then $\gamma_{k+1} > \gamma_k$. By hypothesis we have $L_m(\xi_k) = m_k$. If we write $S_k = \{z_n^k : n \geq 1\}$, then we claim that the sequences $T_k = \{\omega_n^k = L_{z_n^k}(\xi_k) : n \geq 1\}$ satisfy (a), (b) and (c). Observe that since $\xi_0 = 0$ then $\omega_n^0 = L_{z_n^0}(0) = z_n^0$ for all $n \geq 1$, and consequently $T_0 = S_0$. Let $\omega_n^k = L_{z_n^k}(\xi_k)$ and $\omega_p^l = L_{z_p^l}(\xi_l)$. If $z_n^k = z_p^l$ then

$$\begin{aligned} \rho(\omega_n^k, \omega_p^l) &= \rho\big(L_{z_n^k}(\xi_k), L_{z_p^l}(\xi_l)\big) = \rho\big(L_{z_n^k}(\xi_k), L_{z_n^k}(\xi_l)\big) \\ &= \rho(\xi_k, \xi_l) = \rho\big(L_m(\xi_k), L_m(\xi_l)\big) = \rho(m_k, m_l). \end{aligned} \tag{4.3}$$

If $l \leq k$ then $S_k \subset S_l$, and consequently there exists a unique $z_{n'}^l \in S_l$ such that $z_{n'}^l = z_n^k$. If $l > k$ then $S_l \subset S_k$, and then there is at most one $z_{n'}^l \in S_l$ such that $z_{n'}^l = z_n^k$. So, when there is such $z_{n'}^l \in S_l$ equality (4.3) tells us that $\rho(\omega_n^k, \omega_{n'}^l) = \rho(m_k, m_l)$. In particular, if $l = 0$ then $\omega_{n'}^0 = z_{n'}^0 = z_n^k$, and

$$\rho(\omega_n^k, z_n^k) = \rho(\omega_n^k, \omega_{n'}^0) = \rho_k. \tag{4.4}$$

Whether $l \leq k$ or $l > k$, for any point ω_p^l in T_l other than the point $\omega_{n'}^l$ defined above, we have that $z_n^k \neq z_p^l$. Therefore (III) implies that $\rho(z_n^k, z_p^l) > \max\{\gamma_k, \gamma_l\}$. In order to fix notation let us say that $l \leq k$, so the above maximum is γ_k. The same argument works for $l > k$, where the maximum is γ_l. Since the function $\log(1+x)/(1-x)$ increases with x

when $0 < x < 1$, then by (4.2) and (4.4),

$$\log\left(\frac{1+\alpha_k}{1-\alpha_k}\right) + \sum_{1\le j\le k} h(m, m_j) = \log\left(\frac{1+\gamma_k}{1-\gamma_k}\right)$$

$$< \log\left(\frac{1+\rho(z_n^k, z_p^l)}{1-\rho(z_n^k, z_p^l)}\right) = h(z_n^k, z_p^l)$$

$$\le h(z_n^k, \omega_n^k) + h(\omega_n^k, \omega_p^l) + h(\omega_p^l, z_p^l)$$

$$= h(m, m_k) + h(\omega_n^k, \omega_p^l) + h(m_l, m).$$

Hence,

$$\log\left(\frac{1+\alpha_k}{1-\alpha_k}\right) < h(\omega_n^k, \omega_p^l) = \log\left(\frac{1+\rho(\omega_n^k, \omega_p^l)}{1-\rho(\omega_n^k, \omega_p^l)}\right),$$

and consequently $\rho(\omega_n^k, \omega_p^l) > \alpha_k$. This proves (a).

Since $T_0 = S_0$ then by (II) $\delta(T_0) > \beta_0 \ge \delta_0$ and (b) holds for T_0. Since $\rho(\omega_n^k, z_n^k) = \rho_k$ for $n \ge 1$ by (4.4), and $\delta(S_k) > \delta_k > 2\rho_k/(1+\rho_k^2)$ by (II) and (4.1), then Lemma 5.3 of [3, Ch. VII] implies that

$$\delta(T_k) \ge \rho\big(\beta_k, 2\rho_k/(1+\rho_k^2)\big).$$

By (II) the above expression is bounded below by δ_k. So, (b) holds and in particular each T_k is an interpolating sequence.

Since $m \in \overline{S}_k$ then there is a subnet (z_α^k) of the sequence $\{z_n^k\}$ such that $z_\alpha^k \to m$. Then $\omega_\alpha^k = L_{z_\alpha^k}(\xi_k)$ is a subnet of T_k that tends to $L_m(\xi_k) = m_k$. Thus, $m_k \in \overline{T}_k \cap \overline{P(m)}$.

Suppose now that $x \in \overline{T}_k \cap \overline{P(m)}$. Then there is a subnet (ω_α^k) of T_k tending to x. Taking a suitable subnet we can assume that the corresponding subnet (z_α^k) of S_k tends to some point $y \in M(H^\infty)$. If $y \notin \overline{P(m)}$ then $P(y)$ does not meet $\overline{P(m)}$ (see [14, Coro. 2.7]). Since $x = \lim_\alpha \omega_\alpha^k = \lim_\alpha L_{z_\alpha^k}(\xi_k) = L_y(\xi_k) \in P(y)$ then $x \notin \overline{P(m)}$, contradicting our assumption. So, $y \in \overline{P(m)}$ and then $y \in \overline{S}_k \cap \overline{P(m)} = \{m\}$, that is, $y = m$. Hence $\omega_\alpha^k = L_{z_\alpha^k}(\xi_k) \to L_m(\xi_k) = m_k$ and $x = m_k$. ■

In [5, pp. 968–973] it is proved that if $m \in G \setminus \mathrm{D}$ is in a homeomorphic disk and B is an interpolating Blaschke product, then there is an interpolating Blaschke product b so that $b \circ L_m = Bg$, with $|g| \ge \varepsilon\big(\delta(b)\big)$. Lemma 4.2 allows us to give a quantitative generalization of this result.

Theorem 4.3 *Let $m \in G \setminus \mathrm{D}$ such that $P(m)$ is a homeomorphic disk and B be a Blaschke product with simple zeros $\{\xi_k\}_{k\ge 0}$. Let $0 < \beta < 1$. Then for every $k \ge 0$ there is an interpolating Blaschke product b_k with zeros $\{\omega_n^k : n \ge 1\}$ such that*

(1) $Z(b_k) \cap \overline{P(m)} = \{m_k\}$, where $m_k = L_m(\xi_k)$,
(2) $b = \prod_{k\ge 0} b_k$ is a Blaschke product,
(3) $(1 - |\omega_n^k|^2)|b'(\omega_n^k)| \ge \beta(1 - |\xi_k|^2)|B'(\xi_k)|$ for every $\omega_n^k \in Z_D(b_k)$, and
(4) $b \circ L_m(z) = B(z)g(z)$, where $\beta \le |g(z)| \le 1$ for all $z \in \mathrm{D}$.

Proof Let $m' \in P(m)$. Since $L_{m'}(z) = L_m(\tau(z))$, with $\tau\colon \mathrm{D} \to \mathrm{D}$ a suitable Möbius transformation, and the expression $(1-|z|^2)|B'(z)|$ is conformally invariant, a moment of reflection shows that the theorem holds if and only if it holds when m is replaced by any $m' \in P(m)$. In particular, we can assume that $m_0 = m$ (*i.e.*, $\xi_0 = 0$) without loss of generality.

We retain the notation of Lemma 4.2. Let $\{\varepsilon_j\}$ be a sequence so that $\rho(m_j, m) = \rho_j < \varepsilon_j < 1$ and $\prod_{j\geq 0} \varepsilon_j > \beta^{1/2}$. First we choose δ_k between $\beta^{1/2}$ and 1, where $\rho_k/(1+\rho_k^2) < \delta_k < 1$ is close enough to 1 so that $\varepsilon(\delta_k) \geq \varepsilon_k$. Then we choose an increasing sequence $\{\alpha_k\}$ such that $\max\{\rho_k, \eta(\delta_k)\} < \alpha_k < 1$. Let $T_k = \{\omega_n^k : n \geq 1\}$ be the sequence constructed in Lemma 4.2 for these values of δ_k and α_k, and denote by b_k the interpolating Blaschke product with zeros T_k.

Since T_k is interpolating, then $Z(b_k) = \overline{T}_k$ and (1) follows from (c) in Lemma 4.2.

Let $\omega_n^k \in T_k$ be arbitrary. If $j \neq k$, by part (a) of the lemma there is at most one $\omega_{n'}^j \in T_j$ such that $\rho(\omega_{n'}^j, \omega_n^k) = \rho(m_j, m_k)$, while $\rho(\omega_p^j, \omega_n^k) > \max\{\alpha_j, \alpha_k\} \geq \alpha_j > \eta(\delta_j)$ for every $p \neq n'$. Thus, Lemma 3.5 implies that for $j \neq k$,

$$|b_j(\omega_n^k)| \geq \rho(m_j, m_k)\varepsilon(\delta_j) \geq \rho(m_j, m_k)\varepsilon_j. \tag{4.5}$$

Since $\rho(m_j, m_k) = \rho\big(L_m(\xi_j), L_m(\xi_k)\big) = \rho(\xi_j, \xi_k)$, then by (b) of Lemma 4.2 and (4.5),

$$\begin{aligned}(1-|\omega_n^k|^2)|b_k'(\omega_n^k)| \prod_{j:j\neq k} |b_j(\omega_n^k)| &\geq \delta_k \prod_{j:j\neq k} \rho(m_j, m_k) \prod_{j:j\neq k} \varepsilon_j \\ &\geq \beta^{1/2}(1-|\xi_k|^2)|B'(\xi_k)|\beta^{1/2} > 0,\end{aligned}$$

implying that $b = \prod_{k\geq 0} b_k$ converges. Therefore, b is a Blaschke product and the first member in the above inequality is $(1-|\omega_n^k|^2)|b'(\omega_n^k)|$. This states (2) and (3). Analogously, if $k = 0$ and we use inequality (4.5) with $j \geq N+1$, then

$$\prod_{j\geq N+1} |b_j(\omega_n^0)| \geq \prod_{j\geq N+1} \rho_j \prod_{j\geq N+1} \varepsilon_j \to 1$$

uniformly on n when $N \to \infty$. Since $m \in \overline{T}_0$ then

$$\prod_{j\geq N+1} |b_j(m)| \to 1 \quad \text{when} \quad N \to \infty. \tag{4.6}$$

Fix an arbitrary $0 < r < 1$ and let $a_N = \prod_{j\geq N+1} b_j$. Then the S-P inequality says that for every $x \in \overline{\Delta}(m, r) = \{x \in M(H^\infty) : \rho(x, m) \leq r\}$ we have $\rho\big(a_N(x), a_N(m)\big) \leq \rho(x, m) \leq r$. So, by (4.6) and the relation between the pseudohyperbolic and the euclidean metrics [3, p. 3],

$$|a_N(x)| \geq \frac{|a_N(m)| - r}{1 - r|a_N(m)|} \to 1 \quad \text{when} \quad N \to 1.$$

Thus, there is some $N = N(r, \beta)$ such that

$$|a_N(x)| > \beta^{1/2} \quad \text{for all } x \in \overline{\Delta}(m, r). \tag{4.7}$$

Since $b = a_N b_1 \cdots b_N$ and for each k we have $Z(b_k) \cap P(m) = \{m_k\}$, then $Z(b) \cap \overline{\Delta}(m,r) = \{m_k : \rho(m_k, m) \le r\}$ with multiplicity 1. The fact that r is arbitrary proves that $b \circ L_m$ only vanishes at the points $\xi_k = L_m^{-1}(m_k)$ $(k \ge 0)$, and with single multiplicity. This means that $b \circ L_m = Bg$, where $g \in H^\infty$ is zero-free on D. Moreover, by [5, Lemma 1.8] $b_k \circ L_m(z) = \phi_k(x) g_k(z)$, where $\phi_k(z) = (z - \xi_k)(1 - \overline{\xi}_k z)^{-1}$ and $|g_k(z)| \ge \varepsilon(\delta_k) > \varepsilon_k$ for every $z \in$ D and $k \ge 0$. Factorizing b we obtain that

$$Bg = b \circ L_m = (a_N \circ L_m). \prod_{0 \le k \le N} (b_k \circ L_m) = B_N G_N \prod_{0 \le k \le N} \phi_k g_k,$$

where B_N is the Blaschke product with zeros $\{\xi_k : k \ge N+1\}$ and $g = G_N g_0 \cdots g_N$.

Since L_m is an isometry from D onto $P(m)$ then $L_m\big(\overline{\Delta}(0,r)\big) = \overline{\Delta}(m,r)$. So, when $|z| \le r$ and $N = N(r,\beta)$ is the the integer of (4.7) we have

$$\begin{aligned} |g(z)| = |G_N(z)|\,|g_0(z)| \cdots |g_N(z)| &\ge |G_N(z) B_N(z)| \varepsilon_0 \cdots \varepsilon_N \\ &> |a_N\big(L_m(z)\big)| \prod_{j \ge 0} \varepsilon_j \ge \beta^{1/2} \beta^{1/2}. \end{aligned}$$

The first and last members of the above inequality turned out to be independent of N, and since r is arbitrary, the inequality holds for every $z \in$ D. ∎

Corollary 4.4 *Let $m \in G \setminus$ D in a homeomorphic disk and let B be an interpolating Blaschke product. Then for any $0 < \beta < 1$ there is an interpolating Blaschke product b (depending on β) such that $\delta(b) \ge \beta\delta(B)$ and $b \circ L_m = Bg$, where $\beta \le |g(z)| \le 1$ for all $z \in$ D.*

Lemma 4.5 *Let S be an interpolating sequence and let $m \in \overline{S} \setminus S$ be a locally w-thin point. Then there exists a subsequence $T \subset S$ such that $m \in \overline{T}$ and T is locally w-thin at m.*

Proof Let R be an interpolating sequence so that $m \in \overline{R}$ and R is locally w-thin at m. It is clear that for any $0 < \alpha < 1$ the point m is in the closure of both sequences

$$T_\alpha \overset{\text{def}}{=} \{z \in S : \rho(z, R) < \alpha\} \quad \text{and} \quad R_\alpha \overset{\text{def}}{=} \{\omega \in R : \rho(\omega, T_\alpha) < \alpha\}.$$

Using (1.1) as a triangular inequality we see that if α is small enough (depending on $\delta(S)$ and $\delta(R)$) then for each $z \in T_\alpha$ there is only one $\omega \in R_\alpha$ such that $\rho(z,\omega) < \alpha$ and vice versa. Since $m \in \overline{R}_\alpha$ and $R_\alpha \subset R$ then R_α is locally w-thin at m, and then so is T_α by a new application of (1.1). ∎

We need the following result of Treil [15].

***Theorem* (Treil)** *Let $f_1, f_2 \in H^\infty$, $\|f_1\| = \|f_2\| = 1$, such that $|f_1(z)| + |f_2(z)| > \delta > 0$ for all $z \in$ D. Then there is a constant $K(\delta) \ge 1$ and $h \in H^\infty$ such that the function $F = f_1 + h f_2$ satisfies*

$$K(\delta)^{-1} \le |F(z)| \le K(\delta).$$

Clearly, $K(\delta)$ can be chosen to be a nonincreasing function of δ.

Lemma 4.6 *Let $m \in G \setminus \mathbb{D}$ and let $0 < \varepsilon < 1$. There is $0 < \beta < 1$, $\beta = \beta(\varepsilon) \to 1$ when $\varepsilon \to 0$, such that whenever $f \in H^\infty$ satisfies*

$$1 \leq |f(L_m(z))| \leq 1/\beta \quad \text{for all } z \in \mathbb{D}, \tag{4.8}$$

then there exists $f_1 \in f + I_m$ such that

$$1 - \varepsilon \leq |f_1(z)| \leq \frac{1}{\beta}(1+\varepsilon) \quad \text{for all } z \in \mathbb{D}. \tag{4.9}$$

Proof Writing $g = f \circ L_m$, inequality (4.8) yields

$$\beta \leq \frac{|g(z)|}{\|g\|} \leq 1 \quad \text{for all } z \in \mathbb{D}. \tag{4.10}$$

Let N be a positive integer to be chosen later. Let us fix a constant $C > K$, where K is the absolute constant appearing in Theorem 2.2. Since $(f/\|g\|)^N \circ L_m = (g/\|g\|)^N$ then Theorem 2.2 implies that there exists some $f_N \in H^\infty$ such that $f_N \circ L_m = (g/\|g\|)^N$ and $\|f_N\| \leq C$. By (4.10) we also have that $|f_N| \geq \beta^N$ on $\overline{P(m)}$, which together with the above estimate gives $|f_N|/\|f_N\| \geq \beta^N/C$ on $\overline{P(m)}$. Let U be an open neighborhood of $\overline{P(m)}$ so that $\overline{U}$ does not meet the Shilov boundary of H^∞, and

$$|f_N|/\|f_N\| > \beta^{2N}/C \quad \text{on} \quad U.$$

As showed in the proof of Theorem 2.2, under these circumstances there exists some inner function $u \in I_m$ such that

$$|u| \geq \beta^{2N}/C \quad \text{on} \quad M(H^\infty) \setminus U.$$

Thus, $|f_N|/\|f_N\| + |u| \geq \beta^{2N}/C$ on $M(H^\infty)$. If $\beta^{2N} \geq 1/2$ then Treil's theorem tells us that there is some $h \in H^\infty$ so that $F = f_N/\|f_N\| + hu$ satisfies

$$K(1/2C)^{-1} \leq K(\beta^{2N}/C)^{-1} \leq |F(z)| \leq K(\beta^{2N}/C) \leq K(1/2C) \tag{4.11}$$

for all $z \in \mathbb{D}$. Since $hu \in I_m$ then $F \circ L_m = (f_N/\|f_N\|) \circ L_m = (g/\|g\|)^N$. Since F is invertible, it has some N-root $F^{1/N}$, and consequently $(F^{1/N} \circ L_m)^N = (F^{1/N})^N \circ L_m = (g/\|g\|)^N$. So, $F^{1/N} \circ L_m = \lambda g/\|g\|$, where $\lambda \in \mathbb{C}$ is some N-root of the unity. Take $N = N(\varepsilon)$ big enough so that $K(1/2C)^{1/N} \leq 1 + \varepsilon$ and consider $f_1 = \overline{\lambda}\|g\|F^{1/N}$. Since

$$f_1 \circ L_m = \overline{\lambda}\|g\|(F^{1/N} \circ L_m) = g = f \circ L_m,$$

then $f_1 \in f + I_m$. Besides, by (4.11) and our choice of N,

$$\|g\|(1-\varepsilon) \leq \|g\|K(1/2C)^{-1/N} \leq |f_1| \leq \|g\|K(1/2C)^{1/N} \leq (1+\varepsilon)\|g\|.$$

Since by (4.8) $1 \leq \|g\| \leq 1/\beta$, then (4.9) follows. Finally, $\beta(\varepsilon) \to 1$ when $\varepsilon \to 0$, because $\beta \geq \sqrt[2N]{1/2}$ and $N \geq \log K(1/2C)/\log(1+\varepsilon)$. ■

Proof of Theorem 4.1 In the first statement of the theorem we only have to prove that

$$\|f + I_m\|_{H^\infty/I_m} \le \|f \circ L_m\|_\infty. \tag{4.12}$$

Let $\varepsilon > 0$ and let B be a Blaschke product with simple zeros. Let $0 < \beta < 1$ to be chosen later. By Theorem 4.3 there is a Blaschke product b such that $b \circ L_m = Bg$, where $\beta < |g(z)| < 1$, and by Theorem 3.6 there is $G \in H^\infty$ so that $G \circ L_m = g^{-1}$. Thus, $1 < |G(x)| < \beta^{-1}$ for every $x \in P(m)$. Choosing $\beta = \beta(\varepsilon)$ close enough to 1, Lemma 4.6 assures that there is $G_1 \in G + I_m$ (*i.e.*, $G_1 \circ L_m = g^{-1}$) such that $\|G_1\|_\infty \le 1 + \varepsilon$. Writing $f_1 = G_1 b$ we have $f_1 \circ L_m = B$ and $\|f_1\| = \|G_1\| \le 1 + \varepsilon$. In other words, whenever $f \in H^\infty$ is such that $f \circ L_m = B$, then

$$\|f + I_m\|_{H^\infty/I_m} = \inf\{\|k\|_\infty : k \in f + I_m\} \le 1 + \varepsilon \tag{4.13}$$

for every $\varepsilon > 0$. Now suppose that $f \circ L_m = h$, where $\|h\| = 1$. It is easy to see that every Blaschke product can be uniformly approximated by simple Blaschke products, and well known that the closed convex hull of the set of Blaschke products is the unit ball of H^∞ (see [10] or [3, p. 196]). Therefore, there are simple Blaschke products $b_1, \dots, b_n$ and positive real numbers $\lambda_1, \dots, \lambda_n$ such that $\sum_{1\le j\le n} \lambda_j = 1$ and $\|h - \sum_{1\le j\le n} \lambda_j b_j\|_\infty < \varepsilon$. By (4.13) there are $f_1, \dots, f_n \in H^\infty$ such that

$$f_j \circ L_m = b_j \quad \text{and} \quad \|f_j\| \le 1 + \varepsilon$$

for $1 \le j \le n$. If $F = \sum_{1\le j\le n} \lambda_j f_j$ then $\|F\|_\infty \le 1 + \varepsilon$ and

$$\begin{aligned}
\|f + I_m\|_{H^\infty/I_m} &\le \|(f + I_m) - (F + I_m)\|_{H^\infty/I_m} + \|F + I_m\|_{H^\infty/I_m} \\
&\le \|(f - F) + I_m\|_{H^\infty/I_m} + \|F\|_\infty \\
&\le K\|(f - F) \circ L_m\|_\infty + 1 + \varepsilon \\
&= K\Big\|h - \sum_{1\le j\le n} \lambda_j b_j\Big\|_\infty + 1 + \varepsilon \\
&\le 1 + (K + 1)\varepsilon,
\end{aligned}$$

where K is the constant of Theorem 2.2. Since ε is arbitrary then (4.12) follows.

Suppose that m is a locally thin point and let b be a Blaschke product that is locally thin at m. Multiplying b by a constant of modulus 1 if necessary, by (3.1) we can assume that $b = L_m^{-1}$ and that $(h \circ b) \circ L_m = h$ for every h in H^∞. Thus, if $f \circ L_m = h$ then $h \circ b \in f + I_m$ and $\|h \circ b\| = \|h\| = \|f \circ L_m\|$. As said in Section 3, this property characterizes locally thin points.

Suppose now that $f \in H^\infty$ is a norm 1 function such that $f \circ L_m = u$, a non-constant inner function. Taking $\big(f - f(m)\big)/\big(1 - \overline{f(m)} f\big)$ instead of f we can assume that $f(m) = 0$ (*i.e.*, $u(0) = 0$). Since $f|P(m) \not\equiv 0$ (because $f \circ L_m \not\equiv 0$) then a result of Hoffman [7, Thm. 5.3] implies that there is an interpolating Blaschke product b that divides f with $m \in Z(b)$. Since we are assuming that $P(m)$ is a homeomorphic disk then Corollary 3.4 says that m is a locally w-thin point. Thus Lemmas 4.5 and 3.3 imply that the zero sequence of b has a subsequence T such that $\overline{T} \cap \overline{P(m)} = \{m\}$. Let b_1 be the Blaschke product with zero sequence T. Then b_1 is a factor of f and since $Z(b_1) = \overline{T}$ then $Z(b_1) \cap \overline{P(m)} = \{m\}$. Thus $b_1 \circ L_m(z) = zg(z)$, where $g \in H^\infty$ is invertible and $\|g\|_\infty \leq \|b_1\|_\infty = 1$. Since

$$u = f \circ L_m = \big[(f/b_1) \circ L_m\big][b_1 \circ L_m],$$

then $(f/b_1) \circ L_m = u_0 g^{-1}$, where $u_0(z) = u(z)/z$. Hence, $\|g^{-1}\|_\infty \leq \|f\|_\infty = 1$ yielding $|g| \equiv 1$. So, g is a constant of modulus 1, which proves that b_1 is locally thin at m. ■

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