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THE HOMOLOGY OF SINGULAR POLYGON SPACES

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ABSTRACT. Let M_n be the variety of spatial polygons $P = (a_1, a_2, \ldots, a_n)$ whose sides are vectors $a_i \in \mathbf{R}^3$ of length $|a_i| = 1$ $(1 \le i \le n)$, up to motion in $\mathbf{R}^3$. It is known that for odd n, M_n is a smooth manifold, while for even n, M_n has cone-like singular points. For odd n, the rational homology of M_n was determined by Kirwan and Klyachko [6], [9]. The purpose of this paper is to determine the rational homology of M_n for even n. For even n, let $\tilde{M}_n$ be the manifold obtained from M_n by the resolution of the singularities. Then we also determine the integral homology of $\tilde{M}_n$.

1. **Introduction.** Let M_n be the variety of spatial polygons $P = (a_1, a_2, \ldots, a_n)$ whose sides are vectors $a_i \in \mathbf{R}^3$ of length $|a_i| = 1 (1 \le i \le n)$. Two polygons are identified if they differ only by motions in $\mathbf{R}^3$. The sum of the vectors is assumed to be zero:

$$a_1 + a_2 + \cdots + a_n = 0. \tag{1.1}$$

It is known that M_n admits a Kähler structure such that the complex dimension of M_n is $n-3$. For odd n, M_n has no singular points. For even n, $P = (a_1, a_2, \ldots, a_n)$ is a singular point iff all the $a_i (1 \le i \le n)$ lie on a line in $\mathbf{R}^3$ through O [2], [5], [6], [9]. Such singular points are cone-like singularities and have neighborhoods $C(S^{n-3} \times_{S^1} S^{n-3})$, where C denotes the cone and S^1 acts on both copies of S^{n-3} by complex multiplication [6], [9].

For odd n, $H_*(M_n; \mathbf{Q})$, the rational homology of M_n, was determined by Kirwan and Klyachko [6], [9]. Their strategies are different, but both use theorems in symplectic geometry. Unfortunately, their methods cannot apply to M_n for even n, because of the singular points of M_n.

Thus the purposes of this paper are (a) and (b) below. For the rest of this paper, we always assume n to be even, and sometimes set $n = 2m$.

(a) We determine $H_*(M_n; \mathbf{Q})$. Actually we can also determine $H_q(M_n; \mathbf{Z})$ $(q \ge n-2)$.

(b) Let $\tilde{M}_n$ be the manifold obtained from M_n by the resolution of the singularities. That is, for every singular point of M_n, replace $C(S^{n-3} \times_{S^1} S^{n-3})$ by $D^{n-2} \times_{S^1} S^{n-3}$. Then we determine $H_*(\tilde{M}_n; \mathbf{Z})$.

Our results are as follows. For $H_*(M_n; \mathbf{Q})$, we begin by proving the following:

THEOREM A. *The groups $H_q(M_n; \mathbf{Z})$ $(q \ge n-2)$ are given by:*

(i) $H_{2i+1}(M_n; \mathbf{Z}) = 0$ $(i \ge m-1)$.

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(ii) $H_{2i}(M_n;\mathbf{Z}) \cong \mathbf{Z}^{A_{2i}} (i \geq m-1)$ *with* $A_{2i} = \binom{2m-1}{0} + \binom{2m-1}{1} + \cdots + \binom{2m-1}{2m-3-i}$, *where* $n = 2m$, $\binom{a}{b}$ *denotes the binomial coefficient, and* $\mathbf{Z}^{A_{2i}}$ *denotes the* A_{2i}*-fold direct sum of* $\mathbf{Z}$.

Next we determine the groups $H_q(M_n;\mathbf{Q})$ $(1 \leq q \leq n-4)$, which are isomorphic to $H^q(M_n;\mathbf{Q})$. In order to state the result, we define algebras U, V and a map of algebras $\mu\colon U \longrightarrow V$ as follows. Let U be the algebra over $\mathbf{Q}$ generated by $\alpha_1, \ldots, \alpha_{n-1}$ and f, of degree two, subject to the relations $\alpha_i^2 = -f\alpha_i$ for $1 \leq i \leq n-1$:

$$U = \mathbf{Q}[\alpha_1, \ldots, \alpha_{n-1}, f]/(\alpha_i^2 = -f\alpha_i), \quad \deg \alpha_i = \deg f = 2. \tag{1.2}$$

Next we set

$$S = \left\{(\epsilon_1, \ldots, \epsilon_{n-1}); \epsilon_i = \pm 1 \ (1 \leq i \leq n-1), \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} + 1 = 0\right\}. \tag{1.3}$$

Thus S consists of $\binom{2m-1}{m}$-elements. (Recall that $n = 2m$.) For each $(\epsilon_1, \ldots, \epsilon_{n-1}) \in S$, we denote by $\mathbf{Q}[e_{(\epsilon_1,\ldots,\epsilon_{n-1})}]$ a polynomial algebra on *one* generator $e_{(\epsilon_1,\ldots,\epsilon_{n-1})}$ which has degree two. Then we set

$$V = \bigoplus_{(\epsilon_1,\ldots,\epsilon_{n-1})\in S} \mathbf{Q}[e_{(\epsilon_1,\ldots,\epsilon_{n-1})}]. \tag{1.4}$$

Finally we define a map of algebras $\mu\colon U \longrightarrow V$. In order to do so, it suffices to give $\mu(\alpha_i)$ $(1 \leq i \leq n-1)$ and $\mu(f)$.

(i) For $1 \leq i \leq m-1$, we set

$$\mu(\alpha_i) = -\sum_{\{(\epsilon_1,\ldots,\epsilon_{n-1})\in S;\epsilon_i=-1\}} e_{(\epsilon_1,\ldots,\epsilon_{n-1})}.$$

(ii) For $m \leq i \leq 2m-1$, we set

$$\mu(\alpha_i) = -\sum_{\{(\epsilon_1,\ldots,\epsilon_{n-1})\in S;\epsilon_i=+1\}} e_{(\epsilon_1,\ldots,\epsilon_{n-1})}.$$

(iii) We set

$$\mu(f) = \sum_{(\epsilon_1,\ldots,\epsilon_{n-1})\in S} e_{(\epsilon_1,\ldots,\epsilon_{n-1})}.$$

Now $H^q(M_n;\mathbf{Q})$ $(1 \leq q \leq n-4)$ are given by the following:

THEOREM B. *The map* $\mu\colon U \longrightarrow V$ *is a morphism of algebras and one has*

$$H^{2i}(M_n;\mathbf{Q}) \cong \mathrm{Ker}(\mu\colon U^{2i} \longrightarrow V^{2i}) \ (2 \leq 2i \leq n-4),$$
$$H^{2i+1}(M_n;\mathbf{Q}) \cong \mathrm{Coker}(\mu\colon U^{2i} \longrightarrow V^{2i}) \ (1 \leq 2i+1 < n-4),$$

where U^q *denotes the subspace of* U *consisting of elements of degree* q.

Theorems A and B give $H_q(M_n;\mathbf{Q})$ $(q \neq n-3)$. $H_{n-3}(M_n;\mathbf{Q})$ is determined if we give $\chi(M_n)$, the Euler characteristic of M_n. We set $n = 2m$.

THEOREM C [2]. $\chi(M_{2m}) = -2^{2m-2} + \binom{2m}{m}$.

REMARK 1.5. In [2], $\chi(M_{2m})$ is determined by establishing and then solving a recurrence formula for M_{2m}. As this method needs some effort, we give a more direct proof of Theorem C in this paper.

EXAMPLE 1.6. The rational Poincaré polynomials of M_4, M_6 and M_8 are given by:

$$P_{\mathbf{Q}}(M_4, t) = 1 + t^2.$$
$$P_{\mathbf{Q}}(M_6, t) = 1 + t^2 + 5t^3 + 6t^4 + t^6.$$
$$P_{\mathbf{Q}}(M_8, t) = 1 + t^2 + 28t^3 + 8t^4 + 14t^5 + 29t^6 + 8t^8 + t^{10}.$$

Note that $M_4 = S^2$.

As an example, we will show how to determine $P_{\mathbf{Q}}(M_8, t)$ in Example 1.6. First we know $H_q(M_8; \mathbf{Q})$ $(q \geq 6)$ by Theorem A. Next we can determine $H^q(M_8; \mathbf{Q})$ $(q \leq 4)$ by Theorem B. For example, the fact that $H^4(M_8; \mathbf{Q}) = \mathbf{Q}^8$ is proved as follows. By Theorem B, we have that $H^4(M_8; \mathbf{Q}) \cong \mathrm{Ker}(\mu \colon U^4 \longrightarrow V^4)$. By (1.2), a basis of U^4 is $\{\alpha_i\alpha_j \ (1 \leq i < j \leq 7), \alpha_i f \ (1 \leq i \leq 7), f^2\}$, and hence $\dim_{\mathbf{Q}} U^4 = 29$. By (1.4), a basis of V^4 is $\{e^2_{(\epsilon_1,\ldots,\epsilon_7)}; (\epsilon_1, \ldots, \epsilon_7) \in S\}$, and hence $\dim_{\mathbf{Q}} V^4 = 35$. Now, since $\mu(\alpha_i)$ $(1 \leq i \leq 7)$ and $\mu(f)$ are described by the above basis of V^4, we can write $\mu \colon U^4 \longrightarrow V^4$ as a 35×29 matrix. Then it is elementary to prove that $\mathrm{Ker}(\mu \colon U^4 \longrightarrow V^4) \cong \mathbf{Q}^8$. Finally we can determine $H^5(M_8; \mathbf{Q})$ by Theorem C.

REMARK 1.7. In [6], [9], $H_*(M_n; \mathbf{Q})$ is determined for odd n. In particular these groups obey Poincaré duality, and $H_q(M_n; \mathbf{Q}) = 0$ for odd q. But for even n, Example 1.6 shows that we cannot expect Poincaré duality to hold for M_n. Moreover in general, we cannot expect that $H_q(M_n; \mathbf{Q}) = 0$ for odd q.

Finally we give $H_*(\tilde{M}_n; \mathbf{Z})$.

THEOREM D. *$H_*(\tilde{M}_n; \mathbf{Z})$ is a free $\mathbf{Z}$-module and $P_{\mathbf{Q}}(\tilde{M}_n, t)$, the rational Poincaré polynomial of $\tilde{M}_n$, is given by*

$$P_{\mathbf{Q}}(\tilde{M}_n, t) = 1 + nt^2 + \cdots + \left\{ 1 + (n-1) + \binom{n-1}{2} + \cdots + \binom{n-1}{\min(i, n-3-i)} \right\} t^{2i} + \cdots + t^{2n-6}.$$

Thus $\tilde{M}_n$ obeys Poincaré duality as expected.

This paper is organized as follows. In Section 2, we give strategies to prove Theorems A and B. Theorems A, B, C and D are proved in Sections 3, 4, 5 and 6 respectively.

Before we leave this section, we note that we can identify M_n with the moduli space of semistable configurations with respect to the action of $\mathrm{PSL}(2, \mathbf{C})$. And the latter arise naturally in the theory of vector bundles and torsion free sheaves [8], [9]. Thus our main theorems give information on this theory.

In the paper [4], we will prove some new results on the topology of M_n for odd n. For example, we determine $\pi_q(M_n)$ $(q \leq n-3)$, then we describe M_n in the oriented cobordism ring.

2. **Strategies for proofs of Theorems A and B.** We set $\mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbf{R}^3$. Recall that M_n is defined from the space of spatial polygons by the action of the groups of motions in $\mathbf{R}^3$. Thus for $P = (a_1, a_2, \ldots, a_n) \in M_n$, we can always assume that $a_n = \mathbf{e}$. More precisely, we define C_n by

$$C_n = \left\{P = (a_1, a_2, \ldots, a_{n-1}) \in (S^2)^{n-1}; a_1 + a_2 + \cdots + a_{n-1} + \mathbf{e} = 0\right\}. \tag{2.1}$$

Regard S^1 as the subgroup of SO(3) consisting of elements which fix $\mathbf{e}$. Then S^1 acts naturally on C_n, and it is clear that

$$M_n = C_n/S^1. \tag{2.2}$$

$P = (a_1, a_2, \ldots, a_{n-1}) \in C_n$ is a singular point iff $a_i = \pm\mathbf{e}$ $(1 \le i \le n-1)$. By the same argument as in the case of M_n [5], [8], we can prove that the singular points of C_n have neighborhoods $C(S^{n-3} \times S^{n-3})$.

Note that the S^1-action on C_n is semifree, *i.e.*, the set of the singular points is exactly the set of the fixed points, and except at the singular points, S^1 acts freely.

Let $i_n\colon C_n \hookrightarrow (S^2)^{n-1}$ be the inclusion (*cf.* (2.1)). We prove Theorems A and B by the following steps.

STEP 1. First we prove the following proposition.

PROPOSITION 2.3. *$(i_n)_*\colon H_q(C_n; \mathbf{Z}) \longrightarrow H_q\big((S^2)^{n-1}; \mathbf{Z}\big)$ are isomorphisms for $q \le n-2$.*

STEP 2. Let $\bar{C}_n$ be the space obtained from C_n by removing Int $C(S^{n-3} \times S^{n-3})$, the interior of $C(S^{n-3} \times S^{n-3})$, for every singular point. Since C_n has $\binom{2m-1}{m}$ singular points, we have

$$C_n = \bar{C}_n \cup \Big(\bigcup_{\binom{2m-1}{m}} C(S^{n-3} \times S^{n-3})\Big), \tag{2.4}$$

where we set $n = 2m$.

Let $\bar{\imath}_n\colon \bar{C}_n \hookrightarrow C_n$ be the inclusion:

$$\bar{C}_n \xrightarrow{\bar{\imath}_n} C_n \xrightarrow{i_n} (S^2)^{n-1} \tag{2.5}$$

Then we prove that $(i_n \cdot \bar{\imath}_n)_*\colon H_q(\bar{C}_n; \mathbf{Z}) \longrightarrow H_q\big((S^2)^{n-1}; \mathbf{Z}\big)$ are isomorphisms for $q \le n-4$.

STEP 3. By using the Serre spectral sequence of the fibration $\bar{C}_n \longrightarrow \bar{C}_n/S^1 \longrightarrow \mathbf{C}P^\infty$, we calculate $H_q(\bar{C}_n/S^1; \mathbf{Z})$ $(q \le n-4)$ from Step 2.

STEP 4. By using the isomorphisms

$$H_q\big(M_n, \{\text{singular points}\}; \mathbf{Z}\big) \cong H^{2n-6-q}(\bar{C}_n/S^1; \mathbf{Z}), \tag{2.6}$$

we determine $H_q(M_n; \mathbf{Z})$ $(q \ge n-2)$ from Step 3, which is Theorem A.

Next we state the strategies for the proof of Theorem B. Note that if we attach $C(S^{n-3} \times_{S^1} S^{n-3})$ to every boundary component of $\bar{C}_n/S^1$, then we obtain M_n:

$$M_n = \bar{C}_n/S^1 \cup \Big(\bigcup_{\binom{2m-1}{m}} C(S^{n-3} \times_{S^1} S^{n-3}) \Big) \tag{2.7}$$

(*cf.* (2.4)).

STEP 5. From the proof of Step 3, we prove that the ring structure of $H^*(\bar{C}_n/S^1; \mathbf{Q})$ $(* \leq n-4)$ is isomorphic to that of U. Then we identify the ring structure of $H^*(\bigcup_{\binom{2m-1}{m}} S^{n-3} \times_{S^1} S^{n-3}; \mathbf{Q})$ $(* \leq n-4)$ with that of V in a suitable manner.

STEP 6. Consider the cohomology Mayer-Vietoris sequence of the pair $\{\bar{C}_n/S^1, \bigcup_{\binom{2m-1}{m}} C(S^{n-3} \times_{S^1} S^{n-3})\}$ (*cf.* (2.7)). Let $j_n: \bigcup_{\binom{2m-1}{m}} S^{n-3} \times_{S^1} S^{n-3} \hookrightarrow \bar{C}_n/S^1$ be the inclusion. Then we prove that $(j_n)^*: H^q(\bar{C}_n/S^1; \mathbf{Q}) \longrightarrow H^q(\bigcup_{\binom{2m-1}{m}} S^{n-3} \times_{S^1} S^{n-3}; \mathbf{Q})$ $(q \leq n-4)$ is equal to $\mu: U^q \longrightarrow V^q$ in Section 1, where U^q and V^q denote the subspaces of U and V consisting of elements of degree q. Thus Theorem B follows.

3. Proof of Theorem A.

We prove Theorem A by following Steps 1–4 in Section 2.

STEP 1. For Step 1, we need to prove Proposition 2.3. We prove this proposition by the idea of [3]. Recall that we have the inclusion $i_n: C_n \hookrightarrow (S^2)^{n-1}$. We write its complement as A_n. Thus

$$A_n = \Big\{(a_1, \ldots, a_{n-1}) \in (S^2)^{n-1}; a_1 + \cdots + a_{n-1} + \mathbf{e} \neq 0\Big\}. \tag{3.1}$$

We define a function $f_n: A_n \longrightarrow \mathbf{R}$ by

$$f_n(a_1, \ldots, a_{n-1}) = -|a_1 + \cdots + a_{n-1} + \mathbf{e}|^2. \tag{3.2}$$

Concerning f_n, we can prove the following Propositions 3.3 and 3.4 in the same way as in [3]. Since the calculations are easy, we omit the details.

PROPOSITION 3.3. *$(a_1, \ldots, a_{n-1}) \in A_n$ is a critical point of f_n iff $a_i = \pm\mathbf{e}$ $(1 \leq i \leq n-1)$.*

We try to determine the index of $H(f_n)$, the Hessian of f_n, at every critical point. We say a critical point $(a_1, \ldots, a_{n-1})$ is of type (k, l) if $\mathbf{e}$ appears k-times and $-\mathbf{e}$ appears l-times in $(a_1, \ldots, a_{n-1})$, such that $k + l = n-1$. Note that $k - l + 1 \neq 0$ by (3.1). Then we have the following:

PROPOSITION 3.4. *The index of $H(f_n)$ at the critical point of type (k, l) is given by*

$$\begin{cases} 2l & k > l \\ 2(k+1) & k < l-1. \end{cases}$$

We note that $k - l + 1 \neq 0$.

Now we complete the proof of Proposition 2.3. By Proposition 3.4, we see that the index of $H(f_n)$ at every critical point is less than or equal to $n-2$. Thus A_n has the homotopy type of an $(n-2)$-dimensional CW complex. By Poincaré-Lefschetz duality $H_q\big((S^2)^{n-1}, C_n; \mathbf{Z}\big) \cong H^{2n-2-q}(A_n; \mathbf{Z})$, we have $H_q\big((S^2)^{n-1}, C_n; \mathbf{Z}\big) = 0$ $(q \leq n-1)$. Hence Proposition 2.3 follows. ∎

This completes Step 1.

STEP 2. We prove the following:

PROPOSITION 3.5.
(i) $H_{2i}(\bar{C}_{2m}; \mathbf{Z}) \cong \mathbf{Z}^{A_{2i}}$ $(0 \leq i \leq m-2)$ *with* $A_{2i} = \binom{2m-1}{i}$.
(ii) $H_{2i+1}(\bar{C}_{2m}; \mathbf{Z}) = 0$ $(0 \leq i \leq m-3)$.

PROOF. By Proposition 2.3, $(i_n)_*: H_q(C_n; \mathbf{Z}) \longrightarrow H_q\big((S^2)^{n-1}; \mathbf{Z}\big)$ are isomorphisms for $q \leq n-2$. By applying the Mayer-Vietoris argument to the pair $\big(\bar{C}_n, \bigcup_{\binom{2m-1}{m}} C(S^{n-3} \times S^{n-3})\big)$, $(\bar{\imath}_n)_*: H_q(\bar{C}_n; \mathbf{Z}) \longrightarrow H_q(C_n; \mathbf{Z})$ are isomorphisms for $q \leq n-4$. Thus $(i_n \cdot \bar{\imath}_n)_*: H_q(\bar{C}_n; \mathbf{Z}) \longrightarrow H_q\big((S^2)^{n-1}; \mathbf{Z}\big)$ are isomorphisms for $q \leq n-4$. Thus Proposition 3.5 follows. ∎

This completes Step 2.

STEP 3. We prove the following:

PROPOSITION 3.6.
(i) $H_{2i}(\bar{C}_{2m}/S^1; \mathbf{Z}) \cong \mathbf{Z}^{A_{2i}}$ $(0 \leq i \leq m-2)$ *with* $A_{2i} = \binom{2m-1}{0} + \binom{2m-1}{1} + \cdots + \binom{2m-1}{i}$.
(ii) $H_{2i+1}(\bar{C}_{2m}/S^1; \mathbf{Z}) = 0$ $(0 \leq i \leq m-3)$.

PROOF. Consider the Serre spectral sequence of the fibration $\bar{C}_n \longrightarrow \bar{C}_n/S^1 \longrightarrow \mathbf{C}P^\infty$. By Proposition 3.5, for dimensional reasons we have $E_2^{s,t} \cong E_\infty^{s,t}$ $(s+t \leq 2m-4)$. Hence Proposition 3.6 follows. ∎

This completes Step 3.

STEP 4. Since $M_n = \bar{C}_n/S^1 \cup \big(\bigcup_{\binom{2m-1}{m}} C(S^{n-3} \times_{S^1} S^{n-3})\big)$ (*cf.* (2.7)), we have the following isomorphisms:

$$\begin{aligned} H_q\big(M_n, \{\text{singular points}\}; \mathbf{Z}\big) &\cong \tilde{H}_q\big(M_n/\{\text{singular points}\}; \mathbf{Z}\big) \\ &\cong \tilde{H}_q\big(\bar{C}_n/S^1/\partial(\bar{C}_n/S^1); \mathbf{Z}\big) \\ &\cong H_q\big(\bar{C}_n/S^1, \partial(\bar{C}_n/S^1); \mathbf{Z}\big) \\ &\cong H^{2n-6-q}(\bar{C}_n/S^1; \mathbf{Z}), \end{aligned}$$

where $\partial(\bar{C}_n/S^1)$ denotes the boundary of $\bar{C}_n/S^1$, and the fourth isomorphism is Poincaré-Lefschetz duality.

Now Theorem A follows from Proposition 3.6. ∎

4. **Proof of Theorem B.** We prove Theorem B by Steps 5 and 6 in Section 2.

STEP 5. (A) First we give an identification of $H^*(\bigcup_{\binom{2m-1}{m}} S^{n-3} \times_{S^1} S^{n-3}; \mathbf{Q})$ ($* \leq n-4$) with V. Recall that $M_n = \bar{C}_n/S^1 \cup \left(\bigcup_{\binom{2m-1}{m}} C(S^{n-3} \times_{S^1} S^{n-3})\right)$ (*cf.* (2.7)), and every $C(S^{n-3} \times_{S^1} S^{n-3})$ corresponds to a singular point of M_n. A singular point of M_n is represented by some $P = (a_1, a_2, \ldots, a_{n-1}) \in (S^2)^{n-1}$ such that $a_i = \pm\mathbf{e}$ and $a_1 + \cdots + a_{n-1} + \mathbf{e} = 0$ (*cf.* Section 2). Set

$$a_i = \epsilon_i \mathbf{e} \ (1 \leq i \leq n-1). \tag{4.1}$$

Then $\epsilon_i = \pm 1$. Note that $a_1 + \cdots + a_{n-1} + \mathbf{e} = 0$ implies $\epsilon_1 + \cdots + \epsilon_{n-1} + 1 = 0$.

Thus every boundary component of $\bar{C}_n/S^1$ (which is homeomorphic to $S^{n-3} \times_{S^1} S^{n-3}$) is labeled by $(\epsilon_1, \ldots, \epsilon_{n-1})$ such that $\epsilon_1 + \cdots + \epsilon_{n-1} + 1 = 0$. Since $H^2(S^{n-3} \times_{S^1} S^{n-3}; \mathbf{Q}) \cong H^2(\mathbf{C}P^{m-2}; \mathbf{Q})$, we denote the generator of the the left side by $\mathbf{e}_{(\epsilon_1, \ldots, \epsilon_{n-1})}$.

Then it is clear that $H^*(\bigcup_{\binom{2m-1}{m}} S^{n-3} \times_{S^1} S^{n-3}; \mathbf{Q})$ ($* \leq n-4$) is isomorphic to V, where V is defined in Section 1.

(B) Next we give an identification of $H^*(\bar{C}_n/S^1, \mathbf{Q})$ ($* \leq n-4$) with U. First we construct the generators of $H_2(\bar{C}_n/S^1, \mathbf{Q})$, which we denote by $\{h_1, \ldots, h_{n-1}, y\}$.

(i) Construction of $\{h_1, \ldots, h_{n-1}\}$.

The proof of Proposition 3.5 shows that $(i_n \cdot \bar{\imath}_n)_*: H_2(\bar{C}_n; \mathbf{Q}) \longrightarrow H_2\left((S^2)^{n-1}; \mathbf{Q}\right)$ is an isomorphism. Denote the standard generators of $H_2\left((S^2)^{n-1}; \mathbf{Q}\right)$ by $\{\sigma_1, \ldots, \sigma_{n-1}\}$. (More precisely, let $\sigma \in H_2(S^2; \mathbf{Q})$ be the canonical generator. Set $\sigma_i = 1 \times \cdots \times 1 \times \sigma \times 1 \times \cdots \times 1$, where the i-th element is σ.) Then set

$$h_i = (p_n)_*\left((i_n \cdot \bar{\imath}_n)_*\right)^{-1}(\sigma_i), \tag{4.2}$$

where $p_n: \bar{C}_n \longrightarrow \bar{C}_n/S^1$ is the projection (*cf.* (4.4)).

(ii) Construction of y.

Consider the boundary component of $\bar{C}_n/S^1$, which corresponds to $(1, \ldots, 1, -1, \ldots, -1)$, *i.e.*, $(\epsilon_1, \ldots, \epsilon_{n-1})$ such that $\epsilon_i = +1$ $(1 \leq i \leq m-1)$ and $\epsilon_i = -1$ $(m \leq i \leq 2m-1)$. Since $H_2(S^{n-3} \times_{S^1} S^{n-3}; \mathbf{Q}) \cong H_2(\mathbf{C}P^{m-2}; \mathbf{Q})$, we denote the generator of the left side by x (*cf.* the definition of $\mathbf{e}_{(\epsilon_1, \ldots, \epsilon_{n-1})}$).

Let $k: S^{n-3} \times_{S^1} S^{n-3} \hookrightarrow \bar{C}_n/S^1$ be the inclusion, where $S^{n-3} \times_{S^1} S^{n-3}$ denotes the boundary component which corresponds to $(1, \ldots, 1, -1, \ldots, -1)$. Set

$$y = k_*(x) \tag{4.3}$$

(*cf.* (4.4)).

$$\begin{array}{ccccc} & & \bar{C}_n & \xrightarrow{\bar{\imath}_n} & C_n \xrightarrow{i_n} (S^2)^{n-1} \\ & & {\scriptstyle p_n}\big\downarrow & & \\ S^{n-3} \times_{S^1} S^{n-3} & \xrightarrow{k} & \bar{C}_n/S^1 & & \end{array} \tag{4.4}$$

Now it is easy to show that $\{h_1, \ldots, h_{n-1}, y\}$ is a basis of $H_2(\bar{C}_n/S^1; \mathbf{Q})$. By taking the dual basis, we get a basis of $H^2(\bar{C}_n/S^1; \mathbf{Q})$, which we denote by $\{\alpha_1, \ldots, \alpha_{n-1}, f\}$.

Recall that the proof of Proposition 3.5 produces a S^1-equivariant map $i_n \cdot \overline{\iota_n}\colon \bar{C}_n \longrightarrow (S^2)^{n-1}$ which is $(n-4)$-connected. Therefore, the homomorphism

$$(4.5) \qquad H^*_{S^1}\big((S^2)^{n-1}; \mathbf{Q}\big) \xrightarrow{(i_n\cdot\overline{\iota_n})^*} H^*_{S^1}(\bar{C}_n; \mathbf{Q}) \cong H^*(\bar{C}_n/S^1; \mathbf{Q})$$

is an isomorphism for $* \leq n-4$, where $H^*_{S^1}$ denotes equivariant cohomology. Recall that $H^*_{S^1}\big((S^2)^{n-1}; \mathbf{Q}\big)$ was determined by Kirwan [7]. In our choice of generators $\alpha_1, \ldots, \alpha_{n-1}$ and f, the structure of $H^*_{S^1}\big((S^2)^{n-1}; \mathbf{Q}\big)$ together with (4.5) tell us that $H^*(\bar{C}_n/S^1, \mathbf{Q})$ $(* \leq n-4)$ is generated by $\alpha_1, \ldots, \alpha_{n-1}$ and f with the relations $\alpha_i^2 = -f\alpha_i$ $(1 \leq i \leq n-1)$. Hence $H^*(\bar{C}_n/S^1, \mathbf{Q})$ $(* \leq n-4)$ is isomorphic to U.

This completes Step 5.

STEP 6. Consider the Mayer-Vietoris sequence of the pair $\{\bar{C}_n/S^1, \bigcup_{\binom{2m-1}{m}} C(S^{n-3} \times_{S^1} S^{n-3})\}$ (*cf.* (2.7)). Let $j_n\colon \bigcup_{\binom{2m-1}{m}} S^{n-3} \times_{S^1} S^{n-3} \hookrightarrow \bar{C}_n/S^1$ be the inclusion. We need to know $(j_n)^*\colon H^q(\bar{C}_n/S^1; \mathbf{Q}) \longrightarrow H^q(\bigcup_{\binom{2m-1}{m}} S^{n-3} \times_{S^1} S^{n-3}; \mathbf{Q})$ $(q \leq n-4)$.

By Step 5, we can regard $(j_n)^*$ as $(j_n)^*\colon U \longrightarrow V$. In order to describe this homomorphism, it suffices to determine $(j_n)^*(\alpha_i)$ $(1 \leq i \leq n-1)$ and $(j_n)^*(f)$. We recall that $S = \{(\epsilon_1, \ldots, \epsilon_{n-1}); \epsilon_i = \pm 1\ (1 \leq i \leq n-1), \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} + 1 = 0\}$ (*cf.* (1.3)). Note that Theorem B follows from the next result:

PROPOSITION 4.6.

(i) For $1 \leq i \leq m-1$, $(j_n)^*(\alpha_i) = -\Sigma_{\{(\epsilon_1,\ldots,\epsilon_{n-1})\in S; \epsilon_i = -1\}}\, e_{(\epsilon_1,\ldots,\epsilon_{n-1})}$.

(ii) For $m \leq i \leq 2m-1$, $(j_n)^*(\alpha_i) = -\Sigma_{\{(\epsilon_1,\ldots,\epsilon_{n-1})\in S; \epsilon_i = +1\}}\, e_{(\epsilon_1,\ldots,\epsilon_{n-1})}$.

(iii) $(j_n)^*(f) = \Sigma_{(\epsilon_1,\ldots,\epsilon_{n-1})\in S}\, e_{(\epsilon_1,\ldots,\epsilon_{n-1})}$.

PROOF. Instead of proving these formulae, we prove similar formulae in $(S^2)^{n-1}$. More precisely, let S^1 act on $(S^2)^{n-1}$ in the same way as on C_n. $P = (a_1, a_2, \ldots, a_{n-1}) \in (S^2)^{n-1}$ is a fixed point iff $a_i = \pm \mathbf{e}$ $(1 \leq i \leq n-1)$. We remove a small open disc around every fixed point, and denote this space by $\bar{D}_n$. Then we have the following commutative diagram:

$$(4.7) \qquad \begin{array}{ccc} \bar{C}_n & \xrightarrow{i_n\cdot\overline{\iota_n}} & (S^2)^{n-1} \\ & \searrow \quad \nearrow & \\ & \bar{D}_n & \end{array}$$

where all arrows are the inclusions.

By the definition of α_i $(1 \leq i \leq n-1)$, $f \in H^2(\bar{C}_n/S^1; \mathbf{Q})$ and $\mathbf{e}_{(\epsilon_1,\ldots,\epsilon_{n-1})} \in H^2\big(\partial(\bar{C}_n/S^1); \mathbf{Q}\big)$, where $\partial(\bar{C}_n/S^1)$ denotes the boundary of $\bar{C}_n/S^1$, it suffices to prove Proposition 4.6(i)–(iii) in $\bar{D}_n/S^1$. That is, we define α_i' $(1 \leq i \leq n-1)$, $f' \in H^2(\bar{D}_n/S^1; \mathbf{Q})$ and $\mathbf{e}'_{(\epsilon_1,\ldots,\epsilon_{n-1})} \in H^2\big(\partial(\bar{D}_n/S^1); \mathbf{Q}\big)$ in the same way as for $\alpha_i, f, \mathbf{e}_{(\epsilon_1,\ldots,\epsilon_{n-1})}$. Then we can prove that α_i', f', $\mathbf{e}'_{(\epsilon_1,\ldots,\epsilon_{n-1})}$ satisfy Proposition 4.6(i)–(iii), where in this case, we shall substitute the inclusion $j_n\colon \partial(\bar{C}_n/S^1) \hookrightarrow \bar{C}_n/S^1$ in Proposition 4.6 with the inclusion $j_n'\colon \partial(\bar{D}_n/S^1) \hookrightarrow \bar{D}_n/S^1$. (Note that every boundary component of $\bar{D}_n$ is homeomorphic to $\mathbf{C}P^{2m-2}$.)

We summarize the constructions of α_i', f', $\mathbf{e}'_{(\epsilon_1,\dots,\epsilon_{n-1})}$ as follows (*cf.* Step 5 (A) and (B)).

(A′) $\mathbf{e}'_{(\epsilon_1,\dots,\epsilon_{n-1})} \in H^2\big(\partial(\bar{D}_n/S^1);\mathbf{Q}\big)$ is defined to be the generator of $H^2(\mathbf{C}P^{2m-2};\mathbf{Q})$.

(B′) $\alpha_1',\dots,\alpha_{n-1}', f' \in H^2(\bar{D}_n/S^1;\mathbf{Q})$ are defined to be the duals of $\{(p_n')_*(\sigma_1),\dots,(p_n')_*(\sigma_{n-1}), y'\}$, where $p_n'\colon \bar{D}_n \longrightarrow \bar{D}_n/S^1$ denotes the projection (which corresponds to the projection $p_n\colon \bar{C}_n \longrightarrow \bar{C}_n/S^1$ in Step 5 (B)(i)). We shall regard σ_i $(1 \le i \le n-1)$, which are defined in Step 5 (B)(i), as elements of $H_2(\bar{D}_n;\mathbf{Q})$, since $H_2(\bar{D}_n;\mathbf{Q}) \cong H_2\big((S^2)^{n-1};\mathbf{Q}\big)$.

y' is defined in the same way as in (4.3), *i.e.*, $y' = (k')_*(x')$, where $k'\colon \mathbf{C}P^{2m-2} \hookrightarrow \bar{D}_n/S^1$ denotes the inclusion of the boundary component which corresponds to $(1,\dots,1,-1,\dots,-1)$, and $x' \in H_2(\mathbf{C}P^{2m-2};\mathbf{Q})$ denotes the generator (*cf.* (4.7)).

$$
\text{(4.8)}\qquad
\begin{array}{ccc}
 & & \bar{D}_n \\
 & & \big\downarrow p_n' \\
\partial(\bar{D}_n/S^1) & \overset{j_n'}{\hookrightarrow} & \bar{D}_n/S^1 \\
 & \nwarrow \qquad \nearrow k' & \\
 & \mathbf{C}P^{2m-2} &
\end{array}
$$

Denote the dual of $\mathbf{e}'_{(\epsilon_1,\dots,\epsilon_{n-1})} \in H^2\big(\partial(\bar{D}_n/S^1);\mathbf{Q}\big)$ by $v_{(\epsilon_1,\dots,\epsilon_{n-1})} \in H_2\big(\partial(\bar{D}_n/S^1);\mathbf{Q}\big)$. We denote the sequence $(1,\dots,1,-1,\dots,-1)$, which was used in Step 5 (B)(ii), by $(\epsilon_1^0,\dots,\epsilon_{n-1}^0)$.

Recall that we have an inclusion $j_n'\colon \partial(\bar{D}_n/S^1) \hookrightarrow \bar{D}_n/S^1$ (*cf.* (4.8)). Now the following lemma is proved easily from the definitions of $v_{(\epsilon_1,\dots,\epsilon_{n-1})}$, $(p_n')_*(\sigma_1),\dots,(p_n')_*(\sigma_{n-1})$ and y'.

LEMMA 4.9.
$$(j_n')_*(v_{(\epsilon_1,\dots,\epsilon_{n-1})}) = y' + \sum_{1\le s\le n-1} \delta_s\big\{(p_n')_*(\sigma_s)\big\},$$
where $\delta_s = \begin{cases} -1 & \epsilon_s = -\epsilon_s^0 \\ 0 & \epsilon_s = \epsilon_s^0. \end{cases}$

Now by taking the dual of Lemma 4.9 we have Proposition 4.6.

This completes the proof of Theorem B. ∎

5. **Proof of Theorem C.** By Theorem A, we know $H_q(M_n;\mathbf{Q})$ $(q \ge n-2)$. Hence in order to determine $\chi(M_n)$, it suffices to determine $\sum_{q\le n-3}(-1)^q \dim H^q(M_n;\mathbf{Q})$.

Recall that we have an inclusion $i_n\colon \mathcal{C}_n \hookrightarrow (S^2)^{n-1}$. Hence we also have an inclusion $M_n \hookrightarrow (S^2)^{n-1}/S^1$. We assume the truth of the following Propositions 5.1 and 5.2 for the moment. As in the proof of Proposition 2.3 in Section 3 Step 1, we set $A_n = (S^2)^{n-1} - \mathcal{C}_n$.

PROPOSITION 5.1. *For $q \le 2m-3$, we have*

$$H_c^q(A_{2m}/S^1;\mathbf{Q}) \cong \begin{cases} \mathbf{Q}^{A_{2i}} \text{ with } A_{2i} = 2^{2m-1} - \binom{2m-1}{m} & q = 2i+1 \ (1 \le i \le m-2) \\ 0 & q = 2i \ (0 \le i \le m-1) \text{ or } q = 1, \end{cases}$$

where H_c^* denotes cohomology with compact supports.

PROPOSITION 5.2. *$\tilde{H}_*\big((S^2)^N/S^1;\mathbf{Q}\big)$ is given by*

$$\tilde{H}_q\big((S^2)^N/S^1;\mathbf{Q}\big) \cong \begin{cases} \mathbf{Q}^{b_q^N} & q = 2i+1 \ (1 \le i \le N-1) \\ 0 & \textit{otherwise}, \end{cases}$$

where

$$b_q^N = \binom{N-1}{\frac{q-1}{2}} + 2\binom{N-2}{\frac{q-1}{2}} + 2^2\binom{N-3}{\frac{q-1}{2}} + \cdots + 2^{\frac{2N-q-1}{2}}\binom{\frac{q-1}{2}}{\frac{q-1}{2}}.$$

PROOF OF THEOREM C. Recall the long exact sequence of cohomology with compact supports of the pair $\big((S^2)^{2m-1}/S^1, M_{2m}\big)$:

$$\begin{aligned} \cdots \longrightarrow H_c^q(A_{2m}/S^1;\mathbf{Q}) \longrightarrow H^q\big((S^2)^{2m-1}/S^1;\mathbf{Q}\big) \longrightarrow H^q(M_{2m};\mathbf{Q}) \\ \longrightarrow H_c^{q+1}(A_{2m}/S^1;\mathbf{Q}) \longrightarrow \cdots. \end{aligned}$$

Since $H_c^{2m-2}(A_{2m}/S^1;\mathbf{Q}) = 0$ by Proposition 5.1, exactness shows that

$$\begin{aligned} \sum_{q\le 2m-3}(-1)^q \dim H^q(M_{2m};\mathbf{Q}) &= \sum_{q\le 2m-3}(-1)^q \dim H^q\big((S^2)^{2m-1}/S^1;\mathbf{Q}\big) \\ &\quad - \sum_{q\le 2m-3}(-1)^q \dim H_c^q(A_{2m}/S^1;\mathbf{Q}). \end{aligned} \tag{5.3}$$

By Proposition 5.2, we have

$$\begin{aligned} &\sum_{q\le 2m-3}(-1)^q \dim H^q((S^2)^{2m-1}/S^1;\mathbf{Q}) \\ &= 1 - b_3^{2m-1} - b_5^{2m-1} - \cdots - b_{2m-3}^{2m-1} \\ &= \begin{cases} 1 - \left\{\binom{2m-2}{1} + 2\binom{2m-3}{1} + \cdots + 2^{2m-3}\binom{1}{1}\right\} \\ \quad - \left\{\binom{2m-2}{2} + 2\binom{2m-3}{2} + \cdots + 2^{2m-4}\binom{2}{2}\right\} \\ \quad \vdots \\ \quad - \left\{\binom{2m-2}{m-2} + 2\binom{2m-3}{m-2} + \cdots + 2^{m}\binom{m-2}{m-2}\right\}. \end{cases} \end{aligned} \tag{5.4}$$

While by Proposition 5.1, we have

$$\sum_{q\le 2m-3}(-1)^q \dim H_c^q(A_{2m}/S^1;\mathbf{Q}) = -(m-2)\left\{2^{2m-1} - \binom{2m-1}{m}\right\}.$$

Hence by (5.3), we have

$$\begin{aligned} \sum_{q\le 2m-3}(-1)^q \dim H^q(M_{2m};\mathbf{Q}) &= (5.4) + (m-2)\left\{2^{2m-1} - \binom{2m-1}{m}\right\} \\ &= -2^{2m-3} - \frac{m-4}{2}\binom{2m-1}{m}. \end{aligned} \tag{5.5}$$

On the other hand, we have

(5. 6)
$$\sum_{q\geq 2m-2}(-1)^q \dim H^q(M_{2m};\mathbf{Q}) = \sum_{i=0}^{m-2}\left\{\binom{2m-1}{0}+\binom{2m-1}{1}+\cdots+\binom{2m-1}{i}\right\}$$
$$= -2^{2m-3}+\frac{m}{2}\binom{2m-1}{m}$$

by Theorem A.

Now we have
$$\chi(M_{2m}) = (5.5) + (5.6)$$
$$= -2^{2m-2}+\binom{2m}{m}.$$

This completes the proof of Theorem C assuming the truth of Propositions 5.1 and 5.2. ∎

PROOF OF PROPOSITION 5.1. As in the case of C_n, the S^1-action on A_{2m} is semifree (*cf.* Section 2), and the fixed point set Σ is

$$\Sigma = \left\{(a_1,\cdots,a_{n-1})\in (S^2)^{n-1}; a_i = \pm\mathbf{e}\ (1\leq i\leq n-1), a_1+\cdots+a_{n-1}+\mathbf{e}\neq 0\right\},$$

which consists of $\left(2^{2m-1}-\binom{2m-1}{m}\right)$-points. Set

$$B_{2m} = A_{2m}-\Sigma.$$

Recall that A_{2m} has the homotopy type of a $2(m-1)$-dimensional CW complex (*cf.* Proposition 3.4). Hence the Mayer-Vietoris argument gives the following information on $H^q(B_{2m};\mathbf{Q})$ $(q\geq 2m-1)$:

(5. 7)
$$H^q(B_{2m};\mathbf{Q}) \cong \begin{cases}\mathbf{Q}^{A_{2i}} \text{ with } A_{2i} = 2^{2m-1}-\binom{2m-1}{m} & q = 4m-3\\ 0 & 2m-1\leq q\leq 4m-4 \text{ or } q\geq 4m-2.\end{cases}$$

Next, by the Serre spectral sequence of the fiber bundle $S^1\longrightarrow B_{2m}\longrightarrow B_{2m}/S^1$, we have the following information on $H^q(B_{2m}/S^1;\mathbf{Q})$ $(q\geq 2m-1)$ from (5.7):

(5. 8)
$$H^q(B_{2m}/S^1;\mathbf{Q}) \cong \begin{cases}\mathbf{Q}^{A_{2i}} \text{ with } A_{2i} = 2^{2m-1}-\binom{2m-1}{m} & q = 2i (m\leq i\leq 2m-2)\\ 0 & q\geq 2m-1 \text{ and } q\neq 2i (m\leq i\leq 2m-2).\end{cases}$$

Since B_{2m}/S^1 is smooth, we have by Poincaré duality $H_c^q(B_{2m}/S^1;\mathbf{Q}) \cong H_{4m-3-q}(B_{2m}/S^1;\mathbf{Q})$. Hence we have the following information on $H_c^q(B_{2m}/S^1;\mathbf{Q})$ $(q\leq 2m-2)$ from (5.8):

(5. 9)
$$H_c^q(B_{2m}/S^1;\mathbf{Q}) \cong \begin{cases}\mathbf{Q}^{A_{2i}} \text{ with } A_{2i} = 2^{2m-1}-\binom{2m-1}{m} & q = 2i+1\ (0\leq i\leq m-2)\\ 0 & q = 2i\ (0\leq i\leq m-1).\end{cases}$$

Now by using the long exact sequence of cohomology with compact supports of the pair $(A_{2m}/S^1, \Sigma)$:

$$\cdots \longrightarrow H_c^q(B_{2m}/S^1;\mathbf{Q}) \longrightarrow H_c^q(A_{2m}/S^1;\mathbf{Q}) \longrightarrow H^q(\Sigma;\mathbf{Q}) \longrightarrow H_c^{q+1}(B_{2m}/S^1;\mathbf{Q}) \longrightarrow \cdots,$$

we can prove Proposition 5.1.

This completes the proof of Proposition 5.1. ∎

PROOF OF PROPOSITION 5.2. We prove Proposition 5.2 by induction on N. For $P = (a_1, a_2, \dots, a_N) \in (S^2)^N/S^1$, we can assume that $a_1^2 \geq 0$ and $a_1^3 = 0$, where we set $a_1 = \begin{pmatrix} a_1^1 \\ a_1^2 \\ a_1^3 \end{pmatrix}$. More precisely, set

$$S^+ = \left\{ a = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} \in S^2; a^2 \geq 0, a^3 = 0 \right\}.$$

Set $T = S^+ \times (S^2)^{N-1}$ and let S^1 act in the obvious way on the subspaces $\{\mathbf{e}\} \times (S^2)^{N-1}$ and $\{-\mathbf{e}\} \times (S^2)^{N-1}$ of T, where $\mathbf{e}$ is defined in Section 2. Write this equivalence relation on T by $\sim$. Then it is clear that $(S^2)^N/S^1 \cong T/\sim$.

Decompose $T/\sim$ as $L^+ \cup L^-$, where

$$L^+ = \left\{ \begin{pmatrix} a_1^1 \\ a_1^2 \\ a_1^3 \end{pmatrix} \times a_2 \times \cdots \times a_{N-1} \in T/\sim; a_1^1 \geq 0, a_i \in S^2 \ (2 \leq i \leq N-1) \right\}.$$

(L^- is defined similarly.) Since $L^+ \cap L^-$ is homeomorphic to $(S^2)^{N-1}$, and $L^{\pm}$ is homotopically equivalent to $(S^2)^{N-1}/S^1$, we can calculate $\tilde{H}_*\big((S^2)^N/S^1;\mathbf{Q}\big)$ from the Mayer-Vietoris sequence of the pair $\{L^+, L^-\}$ by induction on N.

This completes the proof of Proposition 5.2, and hence also that of Theorem C. ∎

6. **Proof of Theorem D.**

Recall that

$$M_{2m} = \bar{C}_n/S^1 \cup \Big(\bigcup_{\binom{2m-1}{m}} C(S^{n-3} \times_{S^1} S^{n-3}) \Big) \tag{6.1}$$

by (2.7), while by the definition of $\tilde{M}_{2m}$ we have

$$\tilde{M}_{2m} = \bar{C}_n/S^1 \cup \Big(\bigcup_{\binom{2m-1}{m}} D^{n-2} \times_{S^1} S^{n-3} \Big). \tag{6.2}$$

First we prove the following:

PROPOSITION 6.3. *For $q \leq 2m-4$, we have*

$$H_q(\tilde{M}_{2m};\mathbf{Z}) \cong \begin{cases} \mathbf{Z}^{A_{2i}} \text{ with } A_{2i} = \binom{2m-1}{0} + \binom{2m-1}{1} + \cdots + \binom{2m-1}{i} & q = 2i \ (0 \leq i \leq m-2) \\ 0 & q = 2i+1 \ (0 \leq i \leq m-3). \end{cases}$$

PROOF. By using the Serre spectral sequence of the fiber bundle $S^{2m-3} \longrightarrow S^{2m-3} \times_{S^1} S^{2m-3} \longrightarrow \mathbf{C}P^{m-2}$, we can easily prove that $i_*\colon H_q(S^{2m-3} \times_{S^1} S^{2m-3}; \mathbf{Z}) \longrightarrow H_q(D^{2m-2} \times_{S^1} S^{2m-3}; \mathbf{Z})$ are isomorphisms for $q \leq 2m-4$, where $i\colon S^{2m-3} \times_{S^1} S^{2m-3} \hookrightarrow D^{2m-2} \times_{S^1} S^{2m-3}$ denotes the inclusion.

Consider the Mayer-Vietoris sequence of the pair $\{\bar{C}_{2m}/S^1, \bigcup_{\binom{2m-1}{m}} D^{2m-2} \times_{S^1} S^{2m-3}\}$ (*cf.* (6.2)). The above assertion concerning i_* shows that the sequences

$$0 \longrightarrow H_q\Big(\bigcup_{\binom{2m-1}{m}} S^{2m-3} \times_{S^1} S^{2m-3}; \mathbf{Z}\Big) \longrightarrow H_q(\bar{C}_{2m}/S^1; \mathbf{Z}) \oplus H_q\Big(\bigcup_{\binom{2m-1}{m}} D^{2m-2} \times_{S^1} S^{2m-3}; \mathbf{Z}\Big) \longrightarrow H_q(\tilde{M}_{2m}; \mathbf{Z}) \longrightarrow 0$$

are split short exact sequences for $q \leq 2m-4$. Hence $H_q(\tilde{M}_{2m}; \mathbf{Z}) \cong H_q(\bar{C}_{2m}/S^1; \mathbf{Z})$ ($q \leq 2m-4$).

Now Proposition 6.3 follows from Proposition 3.6. ∎

By Proposition 6.3 together with the Poincaré duality and the universal coefficient theorem, we can determine $H_q(\tilde{M}_{2m}; \mathbf{Z})$ ($q \geq 2m-2$). We can also prove the fact that $H_{2m-3}(\tilde{M}_{2m}; \mathbf{Z})$ is torsion-free. Hence in order to complete the proof of Theorem D, we need to prove the following:

LEMMA 6.4. $H_{2m-3}(\tilde{M}_{2m}; \mathbf{Q}) = 0$.

PROOF. By (6.1), we have $\chi(M_{2m}) = \chi(\bar{C}_{2m}/S^1) + \binom{2m-1}{m}$. By (6.2), we have $\chi(\tilde{M}_{2m}) = \chi(\bar{C}_{2m}/S^1) + \binom{2m-1}{m}(m-1)$. Hence by using Theorem C, we have

$$\chi(\tilde{M}_{2m}) = -2^{2m-2} + m\binom{2m-1}{m}. \tag{6.5}$$

On the other hand, our information on $H_q(\tilde{M}_{2m}; \mathbf{Z})$ ($q \neq 2m-3$) tells us that

$$\begin{aligned}\sum_q (-1)^q \dim H_q(\tilde{M}_{2m}; \mathbf{Q}) &= 2\left[\sum_{i=0}^{m-2}\left\{\binom{2m-1}{0} + \binom{2m-1}{1} + \cdots + \binom{2m-1}{i}\right\}\right] - \dim H_{2m-3}(\tilde{M}_{2m}; \mathbf{Q}) \\ &= -2^{2m-2} + m\binom{2m-1}{m} - \dim H_{2m-3}(\tilde{M}_{2m}; \mathbf{Q}).\end{aligned}$$

Hence we have $H_{2m-3}(\tilde{M}_{2m}; \mathbf{Q}) = 0$ by (6.5).

This completes the proof of Lemma 6.4, and hence also that of Theorem D. ∎

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