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STABLE HOMOTOPY THEORY OF SIMPLICIAL PRESHEAVES

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Introduction. Let **C** be an arbitrary Grothendieck site. The purpose of this note is to show that, with the closed model structure on the category **S** Pre(**C**) of simplicial presheaves in hand, it is a relatively simple matter to show that the category **S** $\mathrm{Pre}(\mathbf{C})^{\mathrm{stab}}$ of presheaves of spectra (of simplicial sets) satisfies the axioms for a closed model category, giving rise to a stable homotopy theory for simplicial presheaves. The proof is modelled on the corresponding result for simplicial sets which is given in [**1**], and makes direct use of their Theorem A.7.

This result gives a precise description of the associated stable homotopy category $\mathrm{Ho}(\mathbf{S}\ \mathrm{Pre}(\mathbf{C})\ )^{\mathrm{stab}}$, according to well known results of Quillen [**6**]. One will recall, however, that it is preferable to have several different descriptions of the stable homotopy category, for the construction of smash products and the like. Such results will appear as they are needed; they are probably not difficult. The main theorem of this paper is the start, at least. It is already easily seen that étale K-theory may be recovered from the stable homotopy category

$$\mathrm{Ho}(\mathbf{S}\ \mathrm{Pre}(\mathrm{\acute{e}t}|_S)^{\mathrm{stab}})$$

which is associated to the étale site $\mathrm{\acute{e}t}|_S$ for decent schemes S. This result is proved at the end of this paper.

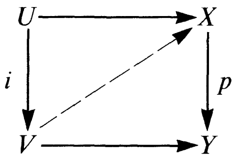
1. Proper closed model categories. We shall begin by recalling some definitions. A *closed model category* is a category $\mathcal{M}$ equipped with three classes of maps, called cofibrations, fibrations, and weak equivalences, which together satisfy the following list of axioms:

CM1. $\mathcal{M}$ is closed under finite direct and inverse limits.

CM2. Given $f:X \rightarrow Y$ and $g: Y \rightarrow Z$, if any two of f, g or $g \circ f$ are weak equivalences, then so is the third.

CM3. If f is a retract of g in the category of arrows of $\mathcal{M}$, and g is a cofibration, fibration or weak equivalence, then so is f.

CM4. Given any commutative solid arrow diagram



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of $\mathscr{M}$, where i is a cofibration and p is a fibration, then the dotted arrow exists making the diagram commute if either i or p is a weak equivalence.

CM5. Any map f of $\mathscr{M}$ may be factored as

(1) $f = p \circ i$, where p is a fibration and i is a cofibration and a weak equivalence.

(2) $f = q \circ j$, where q is a fibration and a weak equivalence and j is a cofibration

A *trivial fibration* is a map which is a fibration and a weak equivalence, and a *trivial cofibration* is a map which is a cofibration and a weak equivalence, as is customary.

A *closed simplicial model category* is a closed model category $\mathscr{M}$ which has a natural function complex $\mathbf{hom}(U, X)$ in the category $\mathbf{S}$ of simplicial sets for each pair of objects U, X in $\mathscr{M}$. This simplicial set is supposed to satisfy some adjointness criteria [**6**, II.1.1], which are never a problem in categories which are constructed from diagrams of simplicial sets. More importantly, $\mathscr{M}$ is to satisfy another axiom:

SM7. If $i:A \rightarrow B$ is a cofibration and $p:X \rightarrow Y$ is a fibration, then the induced map of simplicial sets

$$\mathbf{hom}(B, X) \xrightarrow{(i^*, p_*)} \mathbf{hom}(A, X) \times_{\mathbf{hom}(A, Y)} \mathbf{hom}(B, X)$$

is a Kan fibration, which is trivial if either i or p is trivial. Finally, a closed model category $\mathscr{M}$ is said to be *proper* [**1**] if it satisfies the additional axiom

P. Given a commutative diagram

$$(1.1)\qquad \begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \xrightarrow[g]{} & D \end{array}$$

then,

(1) if the square is a pullback, j is a fibration and g is a weak equivalence, then f is a weak equivalence,

(2) if the square is a pushout, i is a cofibration and f is a weak equivalence, then g is a weak equivalence.

The category $\mathbf{S}$ of simplicial sets, with the usual definitions of fibration, cofibration and weak equivalence, is a proper closed simplicial model category, as is well known [**6**]. P(1) is efficiently proved for $\mathbf{S}$ by replacing the original square, up to weak equivalence, by a cartesian square

$$(1.2)\qquad \begin{array}{ccc} S|A| & \xrightarrow{S|f|} & S|C| \\ {\scriptstyle S|i|}\downarrow & & \downarrow{\scriptstyle S|j|} \\ S|B| & \xrightarrow[S|g|]{} & S|D| \end{array}$$

in which all the objects are Kan complexes, and such that $S|g|$ is a weak equivalence and $S|j|$ is a fibration. As the notation suggests, this is done by realizing and then applying the singular functor; one then uses the exactness properties of realization [**4**, p. 49], and the fact that the realization of a Kan fibration is a Serre fibration [**7**]. The desired result then follows from Brown's "coglueing lemma" [**2**, p. 428]. Axiom P(2) follows from the dual of Brown's result, together with the observation that every simplicial set is cofibrant.

Recall [**5**] that a *cofibration* $f:X \to Y$ of simplicial presheaves is a pointwise monomorphism, meaning that the associated map of sections $f(U):X(U) \to Y(U)$ is a monomorphism of simplicial sets for each object U of the site **C**. A map $g:Z \to W$ is said to be a *weak equivalence* if, for each object U of **C** and each vertex x of $Z(U)$, the induced maps

$$(1.3)\qquad \pi_n(|Z|_U|, x) \overset{f_*}{\to} \pi_n(|W|_U|, fx), \quad n \geqq 0,$$

of presheaves of homotopy groups on the local site $\mathbf{C} \downarrow U$ induce isomorphisms of associated sheaves. If Z and W are presheaves of Kan complexes on **C**, then the topological homotopy groups in the defining condition may be replaced by simplicial homotopy groups. Finally, a *global fibration* is a map which has the right lifting property with respect to all trivial cofibrations.

PROPOSITION 1.4. *With the definitions given above, the category* **S** Pre(**C**) *is a proper closed simplicial model category.*

Proof. The closed simplicial model structure is derived in [**5**]; the explicit statements appear in Theorem 2.3 and Lemma 3.1. Axiom P(1) is proved by replacing the original diagram (1.1) by a diagram of the form (1.2), by using the same trick. (1.2) is then a cartesian diagram in which every object is locally fibrant in the sense of [**5**]. P(1) then follows from Brown's coglueing lemma, together with the theorem of [**5**, p. 30] which asserts that the category of locally fibrant simplicial presheaves on **C** is a category of fibrant objects for a homotopy theory. Axiom P(2) is trivial once again, since every object of **S** Pre(**C**) is cofibrant.

A *presheaf of spectra* X on the site **C** consists of a sequence X^n, $n \geqq 0$, of globally pointed simplicial presheaves, and pointed maps

$$\sigma : S^1 \wedge X^n \to X^{n+1}, \quad n \geqq 0.$$

Here, S^1 is the simplicial set $\Delta^1/\partial\Delta^1$. Any simplicial set K gives rise to a constant simplicial presheaf, also denoted by K, where $K(U) = K$ for all $U \in \mathbf{C}$. Thus, in particular, a globally pointed simplicial presheaf Y is really a simplicial presheaf map $\Delta^0 \to Y$, or equivalently, a simplicial presheaf Y together with a choice of element in the set

$$\varprojlim_{U \in \mathbf{C}} Y(U)_0.$$

Smash products of globally pointed simplicial presheaves are defined pointwise in the obvious way.

A *map of presheaves of spectra* $f: X \to Y$ consists of pointed maps

$$f^n : X^n \to Y^n, \quad n \geqq 0,$$

which respect the structure above in the sense that the diagrams

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ {\scriptstyle 1 \wedge f^n} \downarrow & & \downarrow {\scriptstyle f^{n+1}} \\ S^1 \wedge Y^n & \xrightarrow[\sigma]{} & Y^{n+1} \end{array}$$

commute for $n \geqq 0$. $f: X \to Y$ is said to be a *strict weak equivalence* (resp. *strict fibration*) if each $f^n : X^n \to Y^n$ is a weak equivalence (resp. global fibration) of simplicial presheaves. A map $(f: X \to Y)$ of presheaves of spectra is said to be a *strict cofibration* if the maps

$$f^0 : X^0 \to Y^0,$$

$$X^{n+1} \cup_{(S^1 \wedge X^n)} (S^1 \wedge Y^n) \to Y^{n+1}, \quad n \geqq 0,$$

are cofibrations of simplicial presheaves. Let

$$\mathbf{S}\ \mathrm{Pre}(\mathbf{C})^{\mathrm{strict}}$$

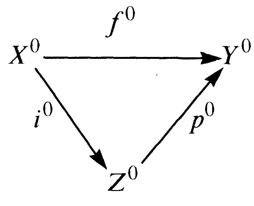
denote the category of presheaves of spectra and maps of such, together with the classes of strict cofibrations, strict fibrations and strict weak equivalences.

PROPOSITION 1.5. *With these definitions, the category*

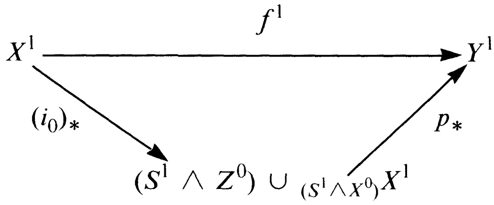
$$\mathbf{S}\ \mathrm{Pre}(\mathbf{C})^{\mathrm{strict}}$$

is a proper closed simplicial model category.

Proof. CM1, CM2 and CM3 are trivial. Suppose that we are given a map $f: X \to Y$. We want to show that f may be factored $f = p \circ i$, where p is a strict fibration and i is a strict cofibration and a strict weak equivalence, thereby proving CM5(1). In effect, find a factorization



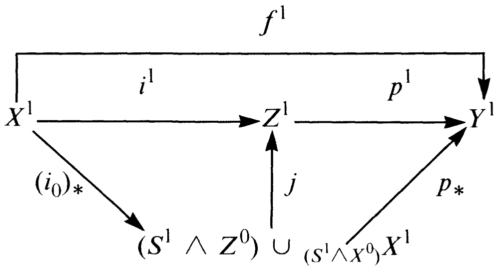
where i^0 is a trivial cofibration of **S** Pre(**C**) and p^0 is a global fibration. Then there is a diagram



where $(i^0)_*$ is a trivial cofibration, since

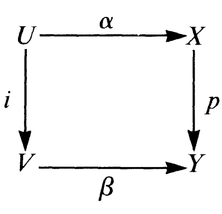
$$1 \wedge i^0 : S^1 \wedge X^0 \to S^1 \wedge Z^0$$

is a trivial cofibration of **S** Pre(**C**) and trivial cofibrations are closed under pushout. But then there is a diagram

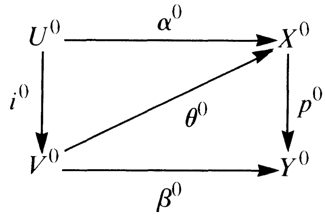


where j is a trivial cofibration and p^1 is a global fibration. Proceed inductively in this manner. CM5(2) is proved similarly.

For CM4, consider the diagram

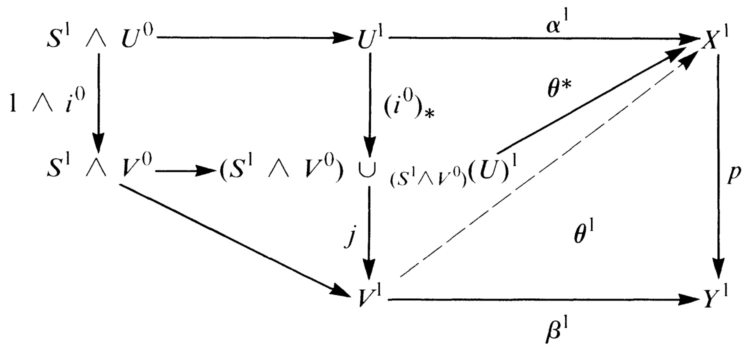


where i is a strict cofibration and p is a trivial strict fibration. Form the diagram



using the closed model structure of **S** Pre(**C**).

Then α^1 and $1 \wedge \theta^0$ induce a diagram



j is a cofibration and p^1 is a trivial global fibration, so the dotted arrow θ^1 exists. Proceed inductively again. The other part of CM4 is similar.

Suppose that X is a presheaf of spectra and that K is a simplicial set. Then one may form the half-smash product $X \ltimes K$ and function object X^K pointwise. Explicitly,

$$\begin{aligned} &(X \ltimes; K)^n(U) \\ &= X(U) \ltimes K = (X(U) \times K)/(* \times K) \quad \text{for } U \in \mathbf{C}, \end{aligned}$$

with structure maps

$$\sigma \ltimes 1 : S^1 \wedge (X^n \ltimes K) \to X^{n+1} \ltimes K.$$

Also, $(X^K)^n(U)$ is the ordinary simplicial set function complex $\mathbf{hom}(K, X^n(U))$, with the map

$$K \to \Delta^0 \xrightarrow{*} X^n(U)$$

as base point. The structure map

$$S^1 \wedge (X^K)^n \to (X^K)^{n+1}$$

is the adjoint of the map

$$(X^K)^n = (X^n)^K \xrightarrow{(\sigma_*)^K} (\Omega X^{n+1})^K \cong \Omega((X^{n+1})^K) = \Omega(X^K)^{n+1}$$

where the isomorphism is canonical and the map

$$\sigma_*: X^n \to \Omega X^{n+1} := \mathbf{hom}_*(S^1, X^n)$$

is the adjoint of

$$\sigma: S^1 \wedge X^n \to X^{n+1}.$$

In particular, given presheaves of spectra U and X, one may form the simplicial set $\mathbf{hom}_*(U, X)$, whose n-simplices are the maps of the form

$$U \ltimes \Delta^n \to X.$$

Finally, if $f: X \to Y$ is a strict fibration (resp. trivial strict fibration) of presheaves of spectra, then

$$X^{\Delta^n} \to X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}, \quad n \geqq 1,$$

is a strict fibration (resp. trivial strict fibration). These statements follow immediately from the fact that $\mathbf{S}$ Pre($\mathbf{C}$) is a closed simplicial model category. Axiom SM7 for

$$\mathbf{S}\ \mathrm{Pre}(\mathbf{C})^{\mathrm{strict}}$$

follows (see [**6**]).

Axiom P for $\mathbf{S}$ Pre($\mathbf{C}$)$^{\mathrm{strict}}$ is a trivial consequence of axiom P for $\mathbf{S}$ Pre($\mathbf{C}$).

2. Stabilization. A presheaf of spectra X is said to be *pointwise fibrant* if each $X^n(U)$, $U \in \mathbf{C}$, $n \geqq 0$, is a Kan complex.

LEMMA 2.1. *There is a functor*

$$X \mapsto X_f$$

from $\mathbf{S}$ Pre($\mathbf{C}$)$^{\mathrm{strict}}$ *to itself, such that X_f is pointwise fibrant. There is a natural strict weak equivalence*

$$\nu_X: X \to X_f.$$

Proof. $X_f^n = S|X^n|$, and $\nu: X^n \to X_f^n$ is the canonical map

$$X^n \to S|X^n|.$$

There is an isomorphism

$$\mathrm{pr}_*: |S^1 \wedge X^n| \xrightarrow{\cong} |S^1| \wedge |X^n|$$

of presheaves of topological spaces which is induced by projections on each factor. There is also a natural map

$$\omega_X: S|S^1| \wedge S|X^n| \to S(|S'| \wedge |X^n|).$$

Define

$$\sigma_f: S^1 \wedge S|X^n| \to S|X^{n+1}|$$

to be the composite

$$S^1 \wedge S|X^n| \xrightarrow{\nu \wedge 1} S|S^1| \wedge S|X^n| \xrightarrow{\omega_X} S(|S^1| \wedge |X^n|)$$

$$\xrightarrow{\cong} S|S^1 \wedge X^n| \xrightarrow{S|\sigma|} S|X^{n+1}|.$$

Then one checks that the diagram

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ {\scriptstyle 1 \wedge \nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ S^1 \wedge S|X^n| & \xrightarrow[\sigma_f]{} & S|X^{n+1}| \end{array}$$

commutes.

Let X be a pointwise fibrant presheaf of spectra. Define

$$QX^n = \varinjlim_i \Omega^i X^{n+i}$$

as a presheaf of simplicial sets, where the map

$$\Omega^i X^{n+i} \to \Omega^{i+1} X^{n+i+1}$$

is induced by applying Ω^i to the map

$$\sigma_* : X^{n+i} \to \Omega X^{n+i+1}.$$

Then QX is a spectrum, with

$$\sigma_* : QX^n \to \Omega QX^{n+1}$$

induced by the diagram

$$\begin{array}{ccccccc} X^n & \xrightarrow{\sigma_*} & \Omega X^{n+1} & \xrightarrow{\Omega\sigma_*} & \Omega^2 X^{n+2} & \longrightarrow & \dots \\ {\scriptstyle \sigma_*}\downarrow & & \downarrow{\scriptstyle \Omega\sigma_*} & & \downarrow{\scriptstyle \Omega^2\sigma_*} & & \\ \Omega X^n & \xrightarrow[\Omega\sigma_*]{} & \Omega^2 X^{n+2} & \xrightarrow[\Omega^2\sigma_*]{} & \Omega^3 X^{n+3} & \longrightarrow & \dots \end{array}$$

There is also a natural map $\tau_X : X \to QX$ which is induced in the obvious way. QX is pointwise fibrant.

An Ω-*object* X is a presheaf of spectra which is pointwise fibrant, and such that each of the maps

$$\sigma_* : X^n \to \Omega X^{n+1}$$

is a weak equivalence of simplicial presheaves. QX is, in particular, an Ω-object, by a cofinality argument. Also, if X is an Ω-object, then the map

$$\tau_X : X \to QX$$

is a strict weak equivalence (see also [**5**, p. 37]). This implies

LEMMA 2.2. *τ_{QX} is a strict weak equivalence for every pointwise fibrant presheaf of spectra X.*

LEMMA 2.3. *$Q\tau_X$ is a strict weak equivalence for every pointwise fibrant presheaf of spectra X.*

Proof. There is a presheaf of spectra ΩX with

$$(\Omega X)^n = \Omega X^{n+1}.$$

Furthermore, there is a commutative diagram

$$\begin{array}{ccccccccc}
\multicolumn{9}{c}{\overset{\tau_X}{\longrightarrow}} \\
X^n & \to & \Omega X^{n+1} & \to & \Omega^2 X^{n+2} & \to & \dots & \to & QX^n \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\Omega X^{n+1} & \to & \Omega^2 X^{n+2} & \to & \dots & & & \to & \Omega Q X^{n+1} \\
\downarrow & & \downarrow & & & & & & \downarrow \\
\Omega^2 X^{n+2} & \to & \Omega^3 X^{n+3} & \to & \dots & & & \to & \Omega^2 X^{n+2} \\
\downarrow & & \downarrow & & & & & & \downarrow \\
\vdots & & \vdots & & & & & & \vdots \\
QX^n & \to & Q(\Omega X)^n & \to & \dots & & & & QQX^n \\
\multicolumn{9}{c}{\underset{Q\tau_X}{\longrightarrow}}
\end{array}$$

All of the maps in the bottom line are weak equivalences, by the same cofinality argument as before.

For arbitrary presheaves of spectra X, let

$$\eta_X : X \to QX_f$$

be the natural map which is defined to be the composite

$$X \xrightarrow{\nu_X} X_f \xrightarrow{\tau_{X_f}} QX_f.$$

LEMMA 2.4. (a) *The functor $X \mapsto QX_f$ preserves strict weak equivalences.*

(b) *$Q(\eta_X)_f$ and η_{QX_f} are strict weak equivalences.*

Proof. (a) is a triviality, since ν_X is a weak equivalence, and Ω^i preserves weak equivalences of presheaves of Kan complexes.

For (b), observe that ν_{QX_f} is a strict weak equivalence, so $(QX_f)_f$ is an Ω-object, and $\tau_{(QX_f)_f}$ is a strict weak equivalence. On the other hand, $Q(\tau_{X_f})$ is a strict weak equivalence by Lemma 2.3. Thus, apply Q to the diagram

$$\begin{array}{ccc} X_f & \xrightarrow{\tau_{X_f}} & QX_f \\ {\scriptstyle \nu_{X_f}}\downarrow & & \downarrow{\scriptstyle \nu_{QX_f}} \\ (X_f)_f & \xrightarrow[(\tau_{X_f})_f]{} & (QX_f)_f \end{array}$$

shows that $Q(\tau_{X_f})_f$ is a strict weak equivalence, and so $Q(\eta_X)_f$ is a strict weak equivalence.

Suppose that X is a pointwise fibrant presheaf of spectra, and consider the system

$$\Omega^{i+n}X^n \to \Omega^{i+n+1}X^{n+1} \to \Omega^{i+n+2}X^{n+2} \to \dots .$$

I define $\tilde{\pi}_i^s(X)$ to be the sheaf of abelian groups which is associated to the presheaf

$$\varinjlim_n \pi_0(\Omega^{i+n}X^n) \cong \varinjlim_n \pi_{i+n}(X^n)$$

where the presheaves of (simplicial) homotopy groups are based at the distinguished base point of X. Recall that there are natural isomorphisms of presheaves

$$\pi_i(X^n) \xrightarrow{(\tau_X)_*} \pi_i(S|X^n|) \overset{\text{adj}}{\cong} \pi_i^{\text{top}}(|X^n|)$$

of presheaves, where $\pi_i^{\text{top}}(|X^n|)$ is the presheaf of topological homotopy groups of the presheaf of spaces $|X^n|$. The isomorphism adj is induced by the adjointness relation between the realization and singular functors.

There is also, of course, a presheaf of spectra of topological spaces $|X|$ associated to such an X. In effect, one realizes the maps

$$\sigma: S^1 \wedge X^n \to X^{n+1},$$

giving maps

$$|S^1| \wedge |X^n| \cong |S^1 \wedge X^n| \overset{|\sigma|}{\to} |X^{n+1}|.$$

The sheaf

$$\tilde{\pi}_i^{s,\text{top}}(|X|)$$

of topological stable homotopy groups is then defined to be the sheaf associated to the presheaf

$$\pi_i^{s,\mathrm{top}}(|X|) = \varinjlim \pi_{i+n}^{\mathrm{top}}(|X^n|),$$

where each map

$$\pi_{i+n}^{\mathrm{top}}(|X^n|) \to \pi_{i+n+1}^{\mathrm{top}}(|X^{n+1}|)$$

is the composite

$$\pi_{i+n}^{\mathrm{top}}(|X^n|) \xrightarrow{|\sigma|_*} \pi_{i+n}^{\mathrm{top}}(\Omega_{\mathrm{top}}|X^{n+1}|) \underset{\cong}{\overset{\partial^{-1}}{\to}} \pi_{i+n+1}^{\mathrm{top}}(|X^{n+1}|).$$

The map

$$|\sigma|_* : |X^n| \to \Omega_{\mathrm{top}}|X^{n+1}|$$

is the adjoint of the map

$$|\sigma| : |S^1| \wedge |X^n| \to |X^{n+1}|,$$

and ∂^{-1} is the map in homotopy groups which is induced by the topological path-loop fibration. $\Omega_{\mathrm{top}}|X^{n+1}|$ is the pointed function space $|X^{n+1}|_*^{|S^1|}$.

LEMMA 2.5. *There is an isomorphism of sheaves*

$$\pi_i^s(X) \cong \tilde{\pi}_i^{s,\mathrm{top}}(|X|)$$

for each pointwise fibrant presheaf of spectra X. This isomorphism is natural in all such X.

Proof. If Y is a fibrant pointed simplicial set and K is a finite pointed simplicial set, then there is a pointed map of topological spaces

$$\omega_{K,X} : |\mathbf{hom}_*(K, X)| \to |X|_*^{|K|}$$

which is natural in X and K. $\omega_{S^0,X}$ is an isomorphism by the Yoneda lemma, since there is a sequence of natural transformations of the form

$$\begin{array}{ccc} \hom_*(Z, \mathbf{hom}_*((K, X))) & \cong & \hom_*(K \wedge Y, Z) \\ \vdots & & \downarrow \\ \vdots & & \hom_*(|K \wedge Y|, |Z|) \\ \vdots & & \downarrow \cong \\ \hom_*(|Z|, |X|_*^{|K|}) & \cong & \hom_*(|K| \wedge |Y|, |Z|) \end{array}$$

In particular, there is a natural comparison of Serre fibre sequences

$$\begin{array}{ccc}
|\Omega K| & \xrightarrow{\omega_{S^1,X}} & \Omega_{\mathrm{top}}|X| \\
\downarrow & & \downarrow \\
|PX| & \longrightarrow & P_{\mathrm{top}}|X| \\
\downarrow & & \downarrow \\
|X| & \xrightarrow[1_{|X|}]{} & |X|
\end{array}$$

[7], and so $\omega_{S^1,X}$ is a natural weak equivalence. It follows that there is a commutative diagram

$$\begin{array}{ccccc}
\pi_{i+n}(X^n) & \xrightarrow{\sigma_*} & \pi_{i+n}(\Omega X^{n+1}) & \overset{\partial^{-1}}{\cong} & \pi_{i+n+1}(X^{n+1}) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\pi_{i+n}^{\mathrm{top}}(\,|X^n|\,) & \xrightarrow{|\sigma_*|} & \pi_{i+n}^{\mathrm{top}}(\,|\Omega X^{n+1}|\,) & \overset{\partial^{-1}}{\cong} & \pi_{i+n+1}^{\mathrm{top}}(\,|X^{n+1}|\,) \\
& \searrow^{|\sigma|_*} & \downarrow \omega_{S^1,X^{n+1}} & & \downarrow = \\
& & \pi_{i+n}^{\mathrm{top}}(\Omega_{\mathrm{top}}|X^{n+1}|\,) & \overset{\partial^{-1}}{\cong} & \pi_{i+n+1}(\,|X^{n+1}|\,)
\end{array}$$

Observe that, by definition,

$$\pi_j QX^n = \pi_{j-n}^s(X)$$

if X is pointwise fibrant. On the other hand, since QX^n is a filtered colimit of presheaves of iterated loop spaces, any local homotopy group sheaf of QX^n is isomorphic to a restriction of $\pi_j QX^n$. This essentially proves

LEMMA 2.6. *A map $f:X \to Y$ of pointwise fibrant presheaves of spectra induces isomorphisms of sheaves*

$$\pi_i^s(X) \cong \pi_i^s(Y), \quad i \in \mathbf{Z},$$

if and only if $Qf:QX \to QY$ is a strict weak equivalence. More generally, a map $g:U \to V$ of presheaves of spectra induces isomorphisms

$$\tilde{\pi}_i^{s,\mathrm{top}}(\,|U|\,) \cong \tilde{\pi}_i^{s,\mathrm{top}}(\,|V|\,), \quad i \in \mathbf{Z},$$

if and only if the induced map

$$Qg_f:QU_f \to QV_f$$

is a strict weak equivalence.

A *Q-weak equivalence* is defined to be a map $g:X \to Y$ of presheaves of spectra which induces a strict weak equivalence

$$Qg_f:QX_f \to QY_f.$$

A *Q-cofibration* is a strict cofibration. A *Q-fibration* is a map of presheaves of spectra which has the right lifting property with respect to all maps which are Q-cofibrations and Q-weak equivalences. $\mathbf{S}\ \mathrm{Pre}(\mathbf{C})^{\mathrm{stab}}$ will refer to the category of presheaves of spectra, together with these three classes of maps.

PROPOSITION 2.7. $\mathbf{S}\ \mathrm{Pre}(\mathbf{C})^{\mathrm{stab}}$ *satisfies Axiom* P.

Proof. To verify P(1), suppose that we are given a cartesian square

$$\begin{array}{ccc} U & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ V & \xrightarrow[g]{} & Y \end{array}$$

in $\mathbf{S}\ \mathrm{Pre}(\mathbf{C})^{\mathrm{stab}}$ such that g is a Q-weak equivalence and p is a Q-fibration. p is a strict fibration, since p has the right lifting property with respect to all maps which are strict cofibrations and strict weak equivalences. Thus, realizing the diagram D gives a comparison of presheaves of fibre sequences

$$\begin{array}{ccc} |F| & = & |F| \\ \downarrow & & \downarrow \\ |U| & \xrightarrow{|f|} & |X| \\ {\scriptstyle |i|}\downarrow & & \downarrow{\scriptstyle |p|} \\ |V| & \xrightarrow[|g|]{} & |Y| \end{array}$$

where F is the fibre of p and $|g|$ induces isomorphisms in all sheaves of stable homotopy groups. But then one compares long exact sequences of sheaves of stable homotopy groups to show that $|f|$ induces an isomorphism in all sheaves of stable homotopy groups.

The proof of P(2) is similar; it uses the natural long exact sequence in stable homotopy groups which is associated to a cofibre sequence of spectra.

THEOREM 2.8. *With the definitions given above,*

$$\mathbf{S}\ \mathrm{Pre}(\mathbf{C})^{\mathrm{stab}}$$

is a proper closed simplicial model category.

Proof. $\mathbf{S}\ \mathrm{Pre}(\mathbf{C})^{\mathrm{stab}}$ is a proper closed model category, by Theorem A.7

of [**1**]. One has to verify their axioms A.4, A.5 and A.6, starting with the stabilization functor $Q(\ )_f$ and the proper closed model structure of

$$\mathbf{S}\ \mathrm{Pre}(\mathbf{C})^{\mathrm{strict}}.$$

But A.4 and A.5 are verified in Lemma 2.4, and A.6 is Proposition 2.7. The function complex $\mathbf{hom}_*(U, X)$ is simply the one described above. SM7 is verified by showing that, if $i: A \to B$ is a Q-trivial cofibration, then so is each induced map

$$(A \ltimes \Delta^n) \cup_{(A \ltimes \partial\Delta^n)} (B \ltimes \partial\Delta^n) \to (B \ltimes \Delta^n), \quad n \geqq 0.$$

In effect, this map is a cofibration, by the SM7 axiom for

$$\mathbf{S}\ \mathrm{Pre}(\mathbf{C})^{\mathrm{strict}},$$

and the functor $A \mapsto A \ltimes K$ preserves Q-trivial cofibrations for each finite simplicial set K, by long exact sequence techniques or otherwise.

The *sphere spectrum* S^0, in this context, is the constant presheaf on the sphere spectrum of the simplicial set category, which consists of the simplicial sets

$$S^0, \quad S^1 \wedge S^0, \quad S^1 \wedge S^1 \wedge S^0, \ldots$$

and the obvious structure maps. For $n \in \mathbf{Z}$, the spectrum $\Sigma^n S^0$ is defined by

$$(\Sigma^n S^0)^k = (S^0)^{n+k}.$$

Observe that S^0 is a Q-cofibrant presheaf of spectra, as are all of its suspensions.

Now suppose that E is a presheaf of spectra on $\mathbf{C}$, and choose a Q-trivial cofibration $i: E \to GE$, where GE is Q-fibrant. Then the group of morphisms $[\Sigma^n S^0, E]$ from $\Sigma^n S^0$ to E in

$$\mathrm{Hom}(\mathbf{S}\ \mathrm{Pre}(\mathbf{C})^{\mathrm{stab}})$$

may be identified with the set

$$\pi(\Sigma^n S^0, GE)$$

of pointed homotopy classes of maps [**6**]. On the other hand, since $\Sigma^n S^0$ is constant, $\pi(\Sigma^n S^0, GE)$ may be identified with the stable homotopy group $\pi_n^s \Gamma_* GE$ of the spectrum $\Gamma_* GE$ of global sections of GE.

A result of Bousfield and Friedlander [**1**] implies that a map $g: X \to Y$ of presheaves of spectra is a Q-fibration if and only if g is a strict fibration and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & QX_f \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Qg_f} \\ Y & \xrightarrow[\eta_Y]{} & QY_f \end{array}$$

is homotopy cartesian. In particular, a presheaf of spectra X is Q-fibrant if and only if X is a strictly fibrant Ω-object. Equivalently, each X^n is a globally fibrant simplicial presheaf, and each map $X^n \to \Omega X^{n+1}$ is a weak equivalence. This means that, if Y is an Ω-object, then one can choose a strict weak equivalence $j: Y \to GY$ from Y to a strictly fibrant presheaf of spectra GY, and then be assured that GY is Q-fibrant.

The mod l K-theory presheaf of spectra

$$K/l = \{K^0/l, K^1/l, \dots\}$$

on the étale site $\text{ét}|_S$ of a scheme S is an Ω-object. This leads, finally, to the observation that the above, via Theorem 3.8 of [**5**], implies that, for decent schemes S and primes l, there are isomorphisms

$$K_i^{\text{ét}}(S; \mathbf{Z}/l) \cong [\Sigma^i S^0, K/l], \quad i \geqq -1.$$

The objects on the left are the étale K-groups of [**3**].

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