Series

1. Prove that the fraction

$$\frac{\prod_{i=1}^{n} 2i-1}{\prod_{i=1}^{n} 2i}$$

when reduced to lowest terms is of the form $a/2^b$ where $0 < b < 2n$.

Solution: We express the fraction as

$$\frac{1}{2^{2n}}\binom{2n}{n}$$

So our only task is to estimate b. The highest power of 2 dividing $\binom{2n}{n}$ is

$$\sum_{i=1}^{\infty}[\frac{2n}{2^i}] - 2\sum_{i=1}^{\infty}[\frac{n}{2^i}].$$

Now,

$$\sum_{i=1}^{\infty}[\frac{2n}{2^i}] = n + \sum_{i=1}^{\infty}[\frac{n}{2^i}],$$

so,

$$\begin{aligned} b &= n + \sum_{i=1}^{\infty}[\frac{n}{2^i}] \\ &< 2n \end{aligned}$$

(1)

The above also shows us that $b > 0$, as required.

2. Let (x_n) be a sequence of numbers satisfying the recurrence:

$$nx_n = (n-3000)x_{n-1} + 1500 \text{ for } n > 0$$

Suppose $x_0 = 0$. Find an expression for x_n for all sufficiently large n.

Solution: Notice, that no matter how the sequence starts $x_{3000} = 1/2$. An easy induction shows that $x_n = 1/2$ for all $n \geq 3000$.

3. Evaluate the sum

$$\sum_{k=1}^{\infty} 3^{k-1} sin^3(x/3^k)$$

Solution: By de Moivre's Theorem, we get the identity

$$sin^3(\theta) = \frac{3}{4}sin(\theta) - \frac{1}{4}sin(3\theta)$$

Using this identity and telescoping we find the j-th partial sum to be:

$$\sum_{k=1}^{j} 3^{k-1} sin^3(x/3^k) = \frac{3^j}{4}sin(x/3^j) - \frac{1}{4}sin(x)$$

Taking limits as $j \to \infty$, we see that the sum is $\frac{x-sin(x)}{4}$.

4. Show that the power-series representation for the series $\sum_{i=0}^{\infty} x^i(x-1)^{2i}/i!$ can't have three consecutive zero coefficients.

Solution: The series sums to $f(x) = e^{x(x-1)^2}$. Let $g(x) = f'(x)/f(x) = 3x^2 - 4x + 1$. By induction, we can prove that for $n \geq 3$,

$$f^{(n+1)}(x) = f^n(x)g(x) + a_n f^{(n-1)}(x)g'(x) + b_n f^{(n-2)}(x)g''(x)$$

for some integers a_n, b_n. Thus, if three consecutive coefficients for the power series were zero, then $f(x)$ would be a polynomial. This is a contradiction.

5. Evaluate the series $\sum_{i=0}^{\infty}(i+1)^2/i!$

Solution: Consider $e^x = \sum_{i=0}^{\infty} x^i/i!$. If we take both sides, multiply by x, differentiate, multiply by x again, and differentiate again, we get the identity

$$(1 + 3x + x^2)e^x = \sum_{i=0}^{\infty}(i+1)^2 x^i/i!.$$

Setting $x = 1$ shows us that the original sum is $5e$.