Pigeon Hole Problems

1. Let u be an irrational real number. Let S be the set of all real numbers of the form $a + bu$, where a and b are integers. Show that S is dense in the real numbers, i.e., for any real number x, and any $\epsilon > 0$, there is any element y in S such that $|x - y| < \epsilon$. (Hint: first try $x = 0$).

Solution: Denote by $\{x\}$ the fractional part of x. We solve the problem for $\epsilon \in (0, 1)$ and $x = 0$. Let N be an integer number such that $N > \frac{1}{\epsilon}$. Consider the numbers $\{0 \cdot u\}, \{1 \cdot u\}, \{2 \cdot u\}, \ldots, \{N \cdot u\}$ and the intervals $[0, \frac{1}{N}), [\frac{1}{N}, \frac{2}{N}), \ldots, [\frac{N-1}{N}, 1)$.

By the pigeonhole principle, there exist p and q between 0 and N such that $|\{pu\} - \{qu\}| < \frac{1}{N}$. Hence, $|(p-q)u + a| < \frac{1}{N}$, where $a = [qu] - [pu]$ is an integer. Let $b = p - q$. Then $|a + bu| < \frac{1}{N} < \epsilon$.

2. If $\mathcal{F}$ is a family of subsets of $[n]$ such that $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$, then $|\mathcal{F}| \leq 2^{n-1}$. Find a family $\mathcal{F}$ of 2^{n-1} subsets that satisfies the previous property.

Solution: Proof by contradiction. Assume that there exists a family $\mathcal{F}$ of subsets of $[n]$ such that $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$ and $|\mathcal{F}| \geq 2^{n-1} + 1$.

Consider the holes $h_A = \{A, [n] \setminus A\}$ with $A \subset [n]$. The number of distinct holes is 2^{n-1}. This is because we have 2^n subsets A of $[n]$ and $h_A = h_{[n]\setminus A}$ for each $A \subset [n]$.

Hence, we have at least $2^{n-1} + 1$ subsets in $\mathcal{F}$ and 2^{n-1} holes h_A. This will lead to a contradiction.

The family $\mathcal{F}_i = \{A \subset [n] : i \in A\}$ has the property that any two subsets in it are intersecting and its size is 2^{n-1}.

3. A lattice point in the plane is a point (x, y) such that both x and y are integers. Find the smallest number n such that given n lattice points in the plane, there exist two whose midpoint is also a lattice point.

Solution: The midpoints of two lattice points (x_1, y_1) and (x_2, y_2) is a lattice point if and only if $x_1 + x_2$ and $y_1 + y_2$ are both even, i.e. x_1 and x_2 have the same parity and y_1 and y_2 have the same parity.

The possible parities of a lattice point coordinates are (even,even),(even,odd),(odd,even) and (odd,odd). By the pigeonhole principle, among any 5 lattice points, there will be two with coordinates of the same parity.

To see that 5 is best possible, consider the points $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$.

4. The points of an infinite rectangular grid are colored with two colours. Show that there are two horizontal and two vertical lines with points at their intersection coloured with the same colour.

Solution: Suppose the colors are red and blue. Consider 9 points on any vertical line. By Pigeon Hole Principle, at least 5 of these have the same color, say red. Now consider the next vertical line on the right and the 5 points to the right of our original five points. At least 3 of these are the same color. If this color is red, we are done. So suppose they are blue. Now consider the 3 points to the right of these three. At least two of these are the same color. Whether these are red or blue, we are done.

5. Prove that there exist integers a, b, c not all zero and each of absolute value less than one million, such that $|a + b(2^{1/2}) + c(3^{1/2})| < 10^{-11}$.

Solution: This is essentially a special case of Problem (1). Again, denote by $\{x\}$ the fractional part of x. Consider the numbers $\{x2^{1/2} + y3^{1/2}\}$ as the integer x ranges from 0 to $4 \cdot 10^5$ and the integer y ranges from $0 to 2.5 \cdot 10^5$. This gives us a few more than 10^11 values in the interval $[0, 1)$. Let $N = 10^{11}$ and break up the interval $[0, 1)$ into intervals $[0, 1/N), [1/N, 2/N)$, etc. Then the Pigeon Hole Principle tells us that at least two of the previous values are in the same interval, say $\{x_0 2^{1/2} + y_0 3^{1/2}\}$ and $\{x_1 2^{1/2} + y_1 3^{1/2}\}$. So

$$|\{x_0 2^{1/2} + y_0 3^{1/2}\} - \{x_1 2^{1/2} + y_1 3^{1/2}\}| < 10^{-11}.$$

Thus setting

$$a = [x_1 2^{1/2} + y_1 3^{1/2}] - [x_0 2^{1/2} + y_0 3^{1/2}], b = x_0 - x_1, c = y_0 - y_1$$

gives the desired result.