WORKING GROUP 16

PREPARATION FOR UNIVERSITY ENGINEERING AND SCIENCE COURSES

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Naima and Richard participated in initial discussions at the Forum in Montreal, and maintained observer status in subsequent conversations, but did not participate in final preparation of this report.

INTRODUCTION

Our working group included 6 university faculty in Mathematics and/or Statistics departments (Robert, Maureen, Ravi, Edgar, Philip, Ross), a Ministry of Education representative from BC (Richard), and a high school teacher (Naima). The composition of the group probably influenced its domain of expertise and is reflected in its findings. (E.g., there were no representatives at all from Quebec or Alberta . . . the two provinces with the best records on such national assessments as the SAIP.)

This document is **not** intended as a description of a grade 12 course, but a description of mathematical expertise accumulated over several courses, with cumulative review and connections. Implementation requires careful attention to sequencing of topics, and reinforcement of skills and understanding.

We identified a three-stage hierarchy of thinking skills, in which students' effort, risk, and reward increase from top to bottom. We expect students to enter University well-equipped with

1. Facts and procedures,
2. Understanding and awareness,
3. Strategies and experience in problem-solving.

FACTS AND PROCEDURES

University instructors expect incoming students to have memorised and mastered certain key facts and algorithms. We spent considerable effort in identifying the facts and algorithms most essential for success in University Science and Engineering. This list does **not** describe the mathematical needs of all high-school mathematics graduates.

Topics have been ordered by importance, but there are many connections, especially between the first three fundamental topics. There is general agreement among group members that topics 1 - 5 (Algebra, Functions, Trigonometry, Vectors, Geometry) are more important. There is some disagreement about the order of importance of topics 6 and 7 (Complex Numbers, Sequences and Series). There is general agreement that topics 8 and 9 (Probability, Statistics) are not as important for students who are about to start a degree in Science or Engineering.

UNDERSTANDING AND AWARENESS

Instant and accurate recall of basic facts and flawless execution of standard procedures would be ideal, but these goals seem unattainable in practice. People are not robots. So it is essential that students' knowledge of facts and procedures be accompanied by enough understanding of their meaning to permit self-correction. University instructors expect incoming students to recognise outrageous answers when they come up (both by recognising the mathematical classification of the desired result and by having some sense of a reasonable value); to recognise correct answers when they come up (without recourse to an outside expert like the answer key); and to be able to "guess" a reasonable answer (again, in both type and value). Students can benefit from explicit instruction not only in how to do things correctly, but also in how to mess things up: what common misconceptions lead to errors, why faulty thinking spoils the result, and how to respond.

Equally important is that students recognise the many connections between the topics outlined below. Students should appreciate that there is often more than one way to tackle a problem. They should also be encouraged to appreciate both the value and the beauty of elegant proofs, and of techniques which generalise readily to more complicated problems.

1.3 STRATEGIES AND EXPERIENCE IN PROBLEM-SOLVING

The purpose of mathematics education in University Science and Engineering is to equip students with enough quantitative insight and strategic flexibility to solve problems that arise outside the classroom or, indeed, the curriculum. This goal should be stated explicitly and pursued relentlessly in every mathematics class from elementary school on. (Some structure is required, of course—there have been spectacular failures of "discovery-based learning" in the early grades where open-ended problems and undefined strategies have robbed the students and their parents of any sense of certainty.) But students who do best in higher mathematics are those who have already learned to make their own problem-solving tools using facts and procedures developed over several topic areas and several years of study.

An important prerequisite skill is the ability to "stick to it". Most interesting problems cannot be solved by quick reference to the most recent worked example in the text or class notes. Students must be exposed to difficult problems, which they cannot solve at the first attempt. Indeed, they must be exposed to failure.

Many students who fail first year university mathematics have difficulty solving multi-stage problems (including familiar problems expressed using unfamiliar stories).

SUPPLEMENTARY INFORMATION

The lists below are extremely terse. Supplementary information is available at
www.math.unb.ca/~maureen/CSMF-Group16

The authors plan to expand the collection of documents available there. It is our hope that the materials will eventually become part of the Forum website, possibly managed by the CMS Education Committee.

1. ALGEBRA

1.1 FACTS AND PROCEDURES

Essential prerequisites: arithmetic with fractions, scientific notation, significant figures, order of magnitude estimates.
Solve systems of (primarily linear) equations in one to three variables.
Solve (primarily linear and quadratic) inequalities in one and two variables; describe solutions using both sets and graphs.
Absolute values; solve equations and inequalities; describe solutions using both sets and graphs.
Powers, radicals; rules for exponents, powers and logs. (These must be "automatic" skills.)
Addition, subtraction and multiplication of polynomials.
Quadratics: factor, complete the square, quadratic formula (prioritised in this order).
Factor: $a^n - b^n$, $a^3 + b^3$, higher degree polynomials that can be factored after changing variables

or spotting a real root.
$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$ (including introduction to factorials).
Long division of polynomials (say, up to degree 5 or 6, divided by degrees 1 and 2).
Factor Theorem.
Remainder Theorem.

1.2 UNDERSTANDING AND AWARENESS
Proper notation: "=", function, sets.
Teachers should set an example by using sentences appropriately.

e.g. "Let x represent the number of / amount of ... ", specifying units where appropriate.

e.g. "The equation $ax^2 + bx + c = 0$ has two solutions, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$." (Rather than simply writing the formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, in isolation, which implies that the symbols a, b, c have special meaning.)
Attention to reversible/irreversible steps in a chain of reasoning.
Cumulative review and assessment – some facts must be remembered for the rest of one's mathematical / scientific career.
When checking numerical examples on a calculator, students should use the calculator's storage and understand order of operations.

1.3 STRATEGIES AND EXPERIENCE IN PROBLEM-SOLVING
For success in university science and engineering, algebraic skills must be as "automatic" as language skills.
Students should always ask themselves:
"Does the answer make sense?"
"Can I check this answer?"
"Are there solutions that I have missed?"
Ratio and proportion problems in applications. (See *Sample Problems* document.)

2. FUNCTIONS

2.1 FACTS AND PROCEDURES
Straight lines: Recognise several forms of an equation whose graph is a straight line (especially the form $Ax + By = C$ and its connection with normal vectors).
Understand the concept "slope".
Find an equation for a line, given slope and one point.
Quadratics: Understand the connection between graph and completed square form.
Polynomials (including quadratic): connection between graph and roots.
Other functions in toolkit: absolute value, polynomials (say, degree up to 5 or 6), rational, trigonometric, log, exponential.
Interpret graphs of functions.
Understand cyclic pattern of graphs of trigonometric functions.
For all the above functions: sketch without a calculator, indicating basic shape, with a few pertinent points plotted accurately.
Recognise graphs of all the above functions.
Connection between solution sets of equations/inequalities and graphs of functions/relations.
Find the equation of a curve, given a few points and information about the class of function (algebraic derivation).
Recognise transformed curves (including completed square form of parabola and
$y = r\cos(\omega t - \phi)$).

Domain and range; composition of functions; inverse functions (with repeated reference to the above functions).

2.2 UNDERSTANDING AND AWARENESS
Goal: that students eventually understand the concept "function". A function is a **rule**, which assigns to each input (often denoted x) a unique output (often denoted y).
Teachers should use function notation rigorously. E.g. The graph of the **function** f defined by the rule $f(x) = x^2 - 4x + 1$ coincides with the graph of **the solution set of the equation** $y + 3 = (x - 2)^2$.
Goal: that students eventually understand composition of functions.
A calculator is a tool for demonstration, investigation, checking answers.
Students are encouraged to see how far they can go without using a calculator.
Encourage the practice of sketching the rough shape of a graph, given its equation.
Describe functions in words. E.g. "The function increase for x in the interval ... It turns around at the point ... ."

2.3 STRATEGIES AND EXPERIENCE IN PROBLEM-SOLVING
Many connections with algebra and trigonometry.

3. TRIGONOMETRY

3.1 FACTS AND PROCEDURES
Pythagorean theorem. Distance formula. Similar triangles.
Definitions of the six trigonometric functions, both for a triangle and on the unit circle.
Trigonometric functions of standard angles: $30°,\ 45°,\ 60°,\ 90°,\ ...\ 180°,\ 210°,\ ...$.
Identities, including:

$$\sin^2\theta + \cos^2\theta = 1, \text{ and identities derived from this.}$$
$$\sin(-\alpha) = -\sin\alpha, \quad \cos(-\alpha) = \cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right), \quad \tan(\alpha + \pi) = \tan\alpha, \quad \text{etc.}$$
$$\sin(\alpha \pm \beta),\ \cos(\alpha \pm \beta),\ \sin(2\alpha),\ \cos(2\alpha)$$

Students should remember the above identities, understand their derivations, and be able to derive new identities.
Understand the cyclic nature of trigonometric functions.
Graph, and recognise graphs of, functions such as $y = r\cos(\omega t - \phi))$ and $y = 2\sin\theta - \cos\theta$.
Inverses of trigonometric functions: understand, manipulate, evaluate with out a calculator.
Law of cosines, law of sines, including connection with formulae $\frac{1}{2}\,bh$ and $\frac{1}{2}\,ab\sin(\theta)$ for area of a triangle.
Degree/radian conversion.

3.2 UNDERSTANDING AND AWARENESS
The cyclic nature of the graphs of trigonometric functions should be emphasised.
Emphasise the three functions $\sin,\ \cos,\ \tan$, with some exposure to $\sec,\ \csc,\ \cot$, which are not as important at this stage.
Students must understand the difference between an identity and an equation (in x, say) that holds for specific values of x.
Connection between identities and graphs of functions.
Connection between solution sets of equations and graphs of functions.

3.3 STRATEGIES AND EXPERIENCE IN PROBLEM-SOLVING
Present students with a trigonometric equation (in angle θ, say), and ask for **all** its solutions.
In order to gain insights (Is this an identity? Is it true for only some θ?), a student may choose to

graph each side of the equation (possibly using a calculator).
Connections can and should be made with: polar coordinates, graphs of curves in parametric form.
See *Sample Problems* document.

4. VECTORS

4.1 FACTS AND PROCEDURES

Addition, subtraction, scalar multiplication.
Geometric interpretation of these.
Equation of plane, related to normal vector.
Connections with trigonometry (cosine law).
3D visualisation.
Connection with $Ax + By = C$ equation of line.

4.2 UNDERSTANDING AND AWARENESS

Encourage visualisation.
Make connections with physics.
Some curricula introduce matrices and matrix multiplication, in which case connections must be made: matrices describe functions; matrix multiplication amounts to composition of functions.
Matrix multiplication provides a convenient review of basic arithmetic.
Once students grasp the idea that a matrix 2×2 describes a special type of function from R^2 to R^2, they will enjoy investigations leading to the discovery that matrices take straight lines to straight lines, triangles to triangles.

4.3 STRATEGIES AND EXPERIENCE IN PROBLEM-SOLVING

It is important that students see matrices with non-integer and non-positive entries. The 2×2 rotation and reflection matrices provide interesting examples.
Matrices and matrix theory are a powerful tool for the description and solution of sytems of equations.

5. GEOMETRY

5.1 FACTS AND PROCEDURES

Synthetic and analytic approaches.
Connections between these.
Facts about triangles.
Similarity (very important).
Opposite angles.
Parallel lines.
Circle geometry – details to be determined.

5.2 UNDERSTANDING AND AWARENESS

Encourage visualisation in 2D and 3D.
Admit that one goal is to practice logical reasoning and argumentation, using a specified set of abstract rules.

5.3 STRATEGIES AND EXPERIENCE IN PROBLEM-SOLVING

Proof, logic; exposure to logical reasoning.

6. COMPLEX NUMBERS

6.1 FACTS AND PROCEDURES
Polar and rectangular forms, and conversion between these forms.
Conjugates.
Addition and subtraction (in rectangular form).
Multiplication, division, powers (in both forms).
Roots, and connection with quadratic and other polynomial equations.
de Moivre's theorem.

6.2 UNDERSTANDING AND AWARENESS
Revisit the quadratic formula.
Many connections with algebra, functions, trigonometry, vectors, geometry.
When using de Moivre's Theorem, students should be encouraged to graph all solutions (and admire the symmetry).

6.3 STRATEGIES AND EXPERIENCE IN PROBLEM-SOLVING
Understanding the symmetry inherent in de Moivre's theorem: students can **sketch** all solutions of (say) $z^{12} = 64$, and hence list them all in polar form.

There are many opportunities for enrichment.

7. SEQUENCES AND SERIES

7.1 FACTS AND PROCEDURES
Finite arithmetic-progressions, and their sums.
Students should understand the connection between arithmetic sequences and functions whose graphs are straight lines.
Finite geometric-progressions, and their sums.
Students should understand the connection between geometric sequences and exponential functions.
Students should establish (prove) formulae for sums of finite arithmetic and geometric series.

7.2 UNDERSTANDING AND AWARENESS
Teachers need be very careful with notation and wording. A sequence is a function whose domain is a set of consecutive integers. For an *infinite* sequence, the domain is usually the set $\{1, 2, 3, ...\}$. For a *finite* sequence, the domain is usually the set $\{1, 2, 3, ..., n\}$, for a specified integer n.

E.g. while the function defined by $y = x^2$, with domain $x \geq 0$ is represented by a smooth half-parabola, the quadratic sequence $a_n = n^2$, $n \geq 0$ is represented by a sequence of dots in the plane, all lying on that half-parabola.
Students should understand that there are **many other** types of sequences, which are neither arithmetic nor geometric.

7.3 STRATEGIES AND EXPERIENCE IN PROBLEM-SOLVING
Some Canadian curricula point out the connection between first differences, second differences, etc and polynomial sequences. This is a fruitful exercise, and connections should be made with graphs of polynomial functions and the slopes of secant lines drawn on those graphs.

Similar remarks apply to common ratios.
Some Canadian curricula relegate financial mathematics for courses designed for non university-bound students. Nevertheless, all students can benefit from interest-rate problems which arise naturally in the setting of arithmetic and geometric series.

8. PROBABILITY

8.1 FACTS AND PROCEDURES
Permutations and combinations.
Conjunction/disjunction.
Proportion.
Binomial distribution (connection with algebra).
Conditional probability (simple cases only).
Independence/dependence.

8.2 UNDERSTANDING AND AWARENESS
The binomial distribution can be derived for small n and demonstrated for larger n (simulation).
Probability is a difficult topic, and teachers must concentrate on simple examples to demonstrate these big ideas.

8.3 STRATEGIES AND EXPERIENCE IN PROBLEM-SOLVING
Careless assignment of problems that are more difficult than they seem can lead to unnecessary student anxiety. At this stage, a few simple examples are sufficient.

9. STATISTICS AND DATA ANALYSIS

9.1 FACTS AND PROCEDURES
Critical reading of newspaper.
Appropriate graphs (create and interpret).
95% confidence interval for proportions – "black box" and simulation.
Connection with Binomial.

9.2 UNDERSTANDING AND AWARENESS
The type of graph is dictated by the type of data. Teachers should be aware that some students have great difficulty recognising different types of data.
Avoid any implication that a "convenience sample" selected for in-class demonstration can be used to make generalisations about a larger population.
Students in science-oriented programs will have mastered many of these skills in other courses.

9.3 STRATEGIES AND EXPERIENCE IN PROBLEM-SOLVING
The biggest hurdle here is students' lack of worldly experience.
It is far easier to discuss other people's statistical studies than to design and analyse one's own experiment. Furthermore, such critical reading helps to develop both reading and analytic skills.
Teachers should collect appropriate newspaper and magazine articles.
The concept of "statistical test" is not appropriate at this stage.

DISCUSSION

Further discussion is needed, and there are some resources we have not yet explored. For example, it would be instructive to compare the list of topics the group identified with the list of topics specified by the National Council of Teachers of Mathematics (NCTM) in its *Principles and Standards for School Mathematics*, published in 2000. The NCTM standards are more ambitious than ours: for example, they endorse complex numbers, vectors, and matrices, which are listed as lower priority topics in this document. It would also be useful to compare our list of topics with those now specified by provincial curricula across Canada. The list of desired topics we have identified is a strict subset of those outlined in curriculum documents for British Columbia and Atlantic Canada, and we believe that the **emphasis** is different than in those documents.

The final draft of this document should be quite specific, so that curriculum designers will understand what is expected.