This book considers inequalities and positivity conditions arising in complex analysis in one and several variables. The methods of proof and applications of these inequalities involve various branches of mathematics. Many of the inequalities are classical; however, the book culminates with some recent research results on positivity conditions for polynomials in several complex variables which are analogs of Hilbert’s 17th problem. The final chapter ties together the earlier material in a very interesting way, and yet the earlier material is much more than simply preparation for the applications in the final chapter.

Chapter 7 gives a complete characterization of bihomogeneous polynomials which are positive when $z \neq 0$, a result of Catlin and D’Angelo and (earlier) Quillen. The
positivity away from the origin of a bihomogeneous polynomial of degree \(2m\) is equivalent to any of the following conditions:

(i) There is an integer \(d\) such that the Hermitian matrix of coefficients for \(||z||^{2d}r(z,\bar{z})\) is positive definite.

(ii) There is an integer \(d\) such that the integral operator on \(A^p(B_n)\) with kernel \(k_d(z,\bar{w}) = <z,w|^d r(z,\bar{w})\) gives a positive definite mapping from \(V_{m+d}\) to itself.

(iii) There is an integer \(d\) and a holomorphic homogeneous vector-valued polynomial \(g\) of degree \(m+d\) such that \(g\) vanishes only at the origin and \(||z||^{2d}r(z,\bar{z}) = ||g(z)||^2\).

(iv) \(r\) is a quotient \(||g||^2/||h||^2\) of squared norms of holomorphic homogeneous polynomial mappings which vanish only at the origin.

Of course this result is reminiscent of Hilbert’s 17th problem. However, the reasons for studying it arise not simply from a desire to generalize but from questions about holomorphic mappings in several complex variables. Another interesting fact is that the theorem of Polya is easily deduced from this result.

Catlin and D’Angelo obtained the characterization (ii) and the other equivalences in 1996 without knowing of Quillen’s work. Their method was to show that the operator with kernel \(r(z,\bar{w}) K(z,\bar{w})\) can be expressed as the sum of a compact operator on \(L^2(B_n)\) and an operator whose restriction to \(A^p(B_n)\) is positive definite; this implies that there is an integer \(j_0\) such that the restriction of the operator to \(V_j \subset A^p(B_n)\) is positive definite for \(j \geq j_0\). Expanding the Bergman kernel function of the unit ball in a series leads to operators with kernels \(k_d(z,\bar{w})\).

Quillen obtained the characterization (i) of positive bihomogeneous polynomials in 1968, also using analytical techniques. He used it to give a proof of the Hilbert Nullstellensatz over \(\mathbb{C}\), noting the analogy with the fact that the Fundamental Theorem of Algebra has an analytical proof using Liouville’s theorem.

In summary, this book is an appealing combination of ideas from linear and polynomial algebra and analysis. It does not go too deeply into technicalities in any one area, but indicates clearly how the study of inequalities leads to basic questions in operator theory, Fourier analysis, several complex variables, and the interaction of linear algebra and polynomial algebra. It is a mathematical smorgasbord with an unexpected treat at the end. Large parts of the book are accessible to a third or fourth year undergraduate; all of it can be read by a beginning graduate student. The idea that different branches of mathematics have deep and important connections is something which it is desirable to communicate to students as soon as possible, in the reviewer’s opinion. This book is very successful at conveying this idea.