This book is an account of a mini-course called *Lectures on Hydrodynamic Scaling* by S.R.S. Varadhan, followed by a workshop on “Hydrodynamic Limits”, both held in October 1998, at the Fields Institute for Research in Mathematical Sciences in Toronto, Canada.

In this report, we mainly focus on Part I of the book, containing the notes by S.R.S. Varadhan. Part II is devoted to papers by the speakers and contributors to the workshop; for each contribution we give the title and author.

**Part I**

Investigation of hydrodynamic limits began in the early 1980’s. However, the two papers that initiated the development of a new theory, by introducing entropy techniques and large deviations to derive hydrodynamic equations, were


In this mini-course of six lectures, S.R.S. Varadhan conveys the main concepts and key technical points of this theory in an enlightening way, thanks to his deep understanding. The example in lecture 1 gives an insight on the issues to cope with. In lectures 2 to 6, through the analysis of an interacting particle system, the simple exclusion process (SEP), are explained gradient and non-gradient tools, and the relative entropy method.

**Lecture 1.** “From classical mechanics to Euler equations”. Quoting S.R.S. Varadhan, *The basic example of hydrodynamical scaling is naturally hydrodynamics itself* (we write quotations in italics). In $\mathbb{R}^3$, $N \approx \bar{\rho} l^3$ classical particles evolve in a periodic cube of side $l$, according to a classical Hamiltonian dynamical system, with 5 conserved quantities (among which the total number $N$ of particles). To each one corresponds a macroscopic quantity: if we rescale space and time by a factor of $l$ (the *hydrodynamic scaling*), is written, then formally derivated, a field (e.g. the density field). Taking limits in $l$ would give a *hydrodynamic equation*. But this requires to replace
microscopic terms appearing in the derivative by their space-time averages w.r.t. equilibrium distributions, through an ergodic theorem, not available for such systems. This leads us to consider stochastic dynamics instead of deterministic ones.

**Lecture 2.** “Some examples”. After the intermediate example of hydrodynamics for noninteracting particles undergoing independent motions, the SEP is introduced. It is an interacting particle system of state space $X = \{0, 1\}^S$, the set of sites $S$ is either $\mathbb{Z}^d$ or $\mathbb{Z}_N^d$ (the quotient obtained from $\mathbb{Z}^d$ by considering each coordinate modulo $N$; we reduce ourselves to this latter case), $N$ will be the scaling parameter. There is at most one particle per site, and for a configuration $\eta \in X$, $\eta(x) = 1$ (resp. $\eta(x) = 0$) means that site $x$ is occupied by a particle (resp. is empty). For the evolution, each site is endowed with a mean one exponential clock, the clocks being mutually independent. When the clock at site $x$ rings, if a particle is present on $x$, it chooses randomly a site $y$ according to a translation invariant probability distribution $p(y - x)$ with finite support, and jumps to that site if $y$ is empty; otherwise nothing happens.

This description corresponds to a Markov process, with infinitesimal generator defined for a local function $f$ (i.e. which depends on a finite number of coordinates) by

$$(Af)(\eta) = \sum_{x,y \in S} \eta(x)(1 - \eta(y))p(y - x)[f(\eta_x^y) - f(\eta)]$$

where $\eta_x^y$ is configuration $\eta$ with occupations at sites $x$ and $y$ exchanged.

Since particles only jump, their total number is a conserved quantity. The Bernoulli product measures $\mu_\rho$, where $\rho \in [0, 1]$ represents the mean density of particles per site, are invariant and translation invariant extremal for the SEP (which is studied in: [L] Interacting Particle Systems, by T.M. Liggett - 1985, Springer).

To go from this microscopic description to a macroscopic one, space is rescaled by $N$ and time by a function $\theta(N)$ ($\theta(N) = N^2$ is the diffusive scaling, $\theta(N) = N$ the hyperbolic one): visualize particles moving on a lattice imbedded in the unit torus $\mathbb{T}^d$, with spacings of $N^{-1}$, that become dense as $N$ tends to $+\infty$; and the generator is $\theta(N)A$. This gives a family of processes denoted by $P_N$ (resp. $P_\rho$) with initial distributions $\mu_N$ (resp. $\mu_\rho$). The question is the macroscopic evolution of density. The stochastic process of empirical distributions

$$\nu_N(t) = \frac{1}{Nd} \sum_{x \in S} \eta(t,x)\delta_{x/N}, \quad 0 \leq t \leq T$$

($\delta_{x/N}$ is the Dirac measure on $x/N$) has values in the space $\mathcal{M}(\mathbb{T}^d)$ of nonnegative measures on $\mathbb{T}^d$, and induced distribution $\hat{P}_N$. To prove the convergence of $\hat{P}_N$ to the Dirac measure concentrated on $\rho(t, r)dr$ where $\rho(., .) \in [0, 1]$ is a weak solution of the hydrodynamic equation, the steps are (1)
the family \( \hat{P} \) is tight, and (2) any limit point \( \hat{P} \) is supported on measure valued maps \( \nu(t) \) weakly continuous in \( t \) with densities \( \rho(t, r) \). For point 2, the following computation shows the problems to overcome, then the methods to proceed are introduced.

For a test function \( J \) on \( \mathbb{T}^d \), write

\[
\xi(t, J) = \xi(0, J) + \int_0^t \theta(N)A(\xi(s, J))ds + M_N(t)
\]

where

\[
\xi(t, J) = \frac{1}{N^d} \sum_{x \in S} J\left( \frac{x}{N} \right) \eta_t(x).
\]

The martingale \( M_N(t) \) is negligible for large \( N \). The difficulty is to replace \( \theta(N)A(\xi(s, J)) \) by a function of \( \xi(s, .) \), to obtain a closed equation for the empirical measure, so that letting \( N \rightarrow \infty \) in (1) will exhibit the hydrodynamic equation.

\[
\theta(N)A(\xi(s, J)) = \frac{1}{N^d} \sum_{x,y \in S} \eta_s(x)(1 - \eta_s(y))p(y - x)\left[ J\left( \frac{y}{N} \right) - J\left( \frac{x}{N} \right) \right]
\]

A Taylor expansion of \( J \), and the properties of \( p \) enable summations by parts. Thus the r.h.s. of (2) is approximated in each of the following cases (we take here \( d = 1 \) to simplify notations) by:

(i) \( p \) is symmetric, i.e. \( p(x) = p(-x) \), \( \theta(N) = N^2 \):

\[
\frac{1}{2} \sum_{x \in S} J'\left( \frac{x}{N} \right) \eta_t(x)
\]

(i') the weakly asymmetric case, where the transition is \( p_N(x) = p(x) + \frac{1}{N} q(x) \), with \( p \) symmetric and \( q(-x) = -q(x) \), \( \theta(N) = N^2 \):

\[
\frac{1}{2} \sum_{x,y \in S} J'\left( \frac{x}{N} \right) J'\left( \frac{y}{N} \right) q(y - x) \times [\eta_s(x)(1 - \eta_s(y)) + \eta_s(y)(1 - \eta_s(x))] \]

(ii) \( p \) has mean zero but is not symmetric, \( \sum_{y \in S} yp(y) = 0 \), \( \theta(N) = N^2 \):

\[
\frac{1}{N} \sum_{x,y \in S} \eta_s(x)\left[ J'\left( \frac{x}{N} \right) + J'\left( \frac{y}{N} \right) \right] \times N(1 - \eta_s(y))(y - x)p(y - x)
\]
\[ = \frac{1}{2N} \sum_{x \in S} J'(x) NW(\tau_x \eta_s) \quad (5) \]

\( \tau_x \) being the translation by \( x \) on the lattice, and

\[ W(\eta) = \eta(0) \sum_{z \in S} (1 - \eta(z))zp(z) - (1 - \eta(0)) \sum_{z \in S} \eta(z)zp(-z) \]

finally (iii) \( \sum_{x,y \in S} yp(y) = m \neq 0, \theta(N) = N \):

\[ \frac{1}{N} \sum_{x,y \in S} J'(\frac{x}{N}) \eta_s(x)N(1 - \eta_s(y))(y - x)p(y - x) \quad (6) \]

In (3), two integrations by parts were possible. This yields via limits in (1) a weak solution of the heat equation

\[ \frac{\partial \rho(t, r)}{\partial t} = \frac{1}{2} \Delta \rho(t, r) \]

with initial condition \( \rho(0, r) = \rho_0(r) \), where \( \Delta \) is the Laplacian. Then, uniqueness of the weak solutions for a given initial density establishes the validity of the hydrodynamic limit.

A way to average the terms in (4) to derive \( \rho(t, r)(1 - \rho(t, r)) \) in the limiting equation is given in

Lecture 3. “Gradient models, Dirichlet forms and large deviations”. The important ingredient in the analysis of gradient models is the ability to do averaging and replace quantities by their expected values calculated under various equilibrium distributions. The main result of [KOV] is an averaging principle that we quote extensively.

**Theorem 3.1.** If \( \mu_N \) is a sequence of probability distributions with density \( f_N \) w.r.t. the uniform distribution, whose Dirichlet form satisfies

\[ D_N(\mu_N) = \frac{1}{2N^d} \sum_{\eta \in X} \sum_{x,y \in S, |x-y|=1} \left[ \sqrt{f_N(\eta^{x,y})} - \sqrt{f_N(\eta)} \right]^2 \leq CN^{d-2} \]

for a constant \( C \) independent of \( N \), then for every local function \( g \),

\[ \lim_{\varepsilon \to 0} \lim \sup_{N \to \infty} \delta_{N,\varepsilon,N} = 0 \]

with

\[ \delta_{N,\ell} = E^{\mu_N} \left[ \frac{1}{N^d} \sum_{x \in S} (Av_\ell g)(\tau_x \eta) - E^{P_\varepsilon}[g(Av_\ell(\eta(x)))] \right]. \]
\[(Av_{\ell}g)(\eta) = \frac{1}{(2\ell + 1)^d} \sum_{z \in B_{\ell}} g(\tau_z \eta); \]

\[B_{\ell} = \{ z \in S : |z_j| \leq \ell; 1 \leq j \leq d \}.\]

The proof of Theorem 3.1 is composed of the one-block estimate:

\[\lim_{\ell \to \infty} \limsup_{N \to \infty} \delta_{N, \ell} = 0,\]

and the two-blocks estimate: *the local density does not fluctuate over small macroscopic length scales*, i.e.

\[\lim_{\varepsilon \to 0, \ell \to \infty} E^{\mu_N} \left[ \frac{1}{N^d} \sum_{x \in S} |Av_{\ell}(\eta(x)) - Av_{N\varepsilon}(\eta(x))| \right].\]

Theorem 3.1 induces superexponential estimates, which imply as Application 1 hydrodynamics in case (i'), and as Application 2 large deviations for the symmetric SEP.

**Lecture 4**, “Relative entropy method”, is devoted to \(d = 1, p(1) = 1, m = 1\), thus deals with (6), where an hyperbolic scaling leads to Burgers equation with zero viscosity

\[\frac{\partial \rho(t, r)}{\partial t} + \frac{\partial [\rho(t, r)(1 - \rho(t, r))]}{\partial r} = 0\]

This method needs both special random initial conditions and a smooth solution \(\rho(t, r)\) of the equation in \([0, T] \times \mathbb{T}\), and it implies uniqueness of that smooth solution. Assume that for some \(c > 0, c \leq \rho(t, r) \leq 1 - c\). For any \(t \in [0, T]\), define a distribution \(\alpha_N(t)\) by \(\alpha_N(t)\{\eta(x) = 1\} = \rho(t, x/N)\); but the actual distribution of the particle system solves the Kolmogorov forward equation \(\frac{d\mu_N(t)}{dt} = A_N^* \mu_N(t)\), with \(\mu_N(0) = \alpha_N(0)\). The latter equation combined with entropy inequality, one-block estimate and a careful reduction to Gronwall’s lemma yield that the relative entropy satisfies

\[\lim_{N \to \infty} \sup_{0 \leq t \leq T} \frac{H(\mu_N(t); \alpha_N(t))}{N} = \lim_{N \to \infty} \sup_{0 \leq t \leq T} \frac{1}{N} \sum_{\eta \in \mathcal{X}} \log \left[ \frac{\mu_N(t, \eta)}{\alpha_N(t, \eta)} \right] \mu_N(t, \eta) = 0.\]

With large deviations estimates, this implies the validity of the hydrodynamic scaling limit.

**Lecture 5**, “Nongradient systems”, is devoted to the asymmetric mean zero case with \(d = 1\), so deals with (5). “Nongradient” means that, contrary to (3), we cannot write the current \(W(\eta)\) as the gradient of a function, and, to derive the hydrodynamic equation

\[\frac{\partial \rho(t, r)}{\partial t} = \frac{1}{2} \frac{\partial}{\partial r} \left( \rho(t, r)^2 \right) \frac{\partial \rho(t, r)}{\partial r}\]
we need a theorem to the effect that there is a function $a(x)$ and local functions $g_x(y)$ such that it is almost true that $W = \mathcal{A}g_x + a(x)(y(0) - y(1))$ relative to $P_x$. This is Theorem 5.5. We do not say more about that case, conceptually and technically much more difficult than the preceding ones.

Lecture 6, “Nongradient case (continuation)”, contains the proof of Theorem 5.5.

To conclude, for an extensive study of the theory, we recommend the book: [KL] Scaling Limits of Interacting Particle Systems, by C. Kipnis, C. Landim (1999, Springer). It is self-contained (the necessary probabilistic and analytical tools are in the appendices), and each result or computation is detailed. We also mention a text whose spirit is in between this mini-course and [KL]: [JY] Jensen L., Yau H.T., Hydrodynamical Scaling Limits of Simple Exclusion Models. IAS/Park City Mathematics Series 6 (1999), 167–225.

Part II

On a 1-D Granular Media Immersed in a Fluid, by J. A. Carrillo.

A Class of Cellular Automata Equivalent to Deterministic Particle Systems, by H. Fukš.

Recent Results on the Ginzburg-Landau $\phi$ Interface Model, by T. Funaki.

Large Scale Behavior of a System of Interacting Diffusions, by I. Grigorescu.


Free Boundary Problem and Hydrodynamic Limit, by J. Quastel.


Quantum Mechanics, Linear Boltzmann Equation and Renormalization, by H. T. Yau.