Open Algebraic Surfaces

Book Review by Peter Russell, McGill University

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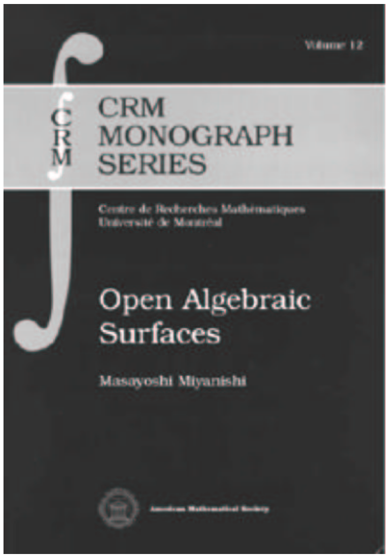

To put the subject matter of “Open Algebraic Surfaces” in perspective, let me begin with a very classical **question**: *If* $k \subset L$ *are fields and* $L \subset k(x_1, \cdots, x_n) = k^{(n)}$ *(field of rational functions), is* L *purely transcendental over* k? (Then $L = k^{(d)}$, $d =$ transcendence degree of L over k. This is the “right” measure for the dimension of the problem, one can easily reduce it to the case $n = d$.) The answer is *yes* if $d = 1$ by **Lüroth’s Theorem**. For $d = 2$ already we have to be more circumspect. The answer is *yes for* k *algebraically closed of characteristic* 0 by **Castelnuovo’s Theorem**. (If char $k > 0$, separability of $k^{(n)}/L$ is required.) The answer is *no* for $d \geq 3$. This is a celebrated result of the 1970’s [CG]. Lüroth’s theorem can be settled by an elementary algebraic argument. For $d \geq 2$, though, our question appears to require a determined plunge into algebraic geometry.

It is tempting, but was done in a systematic way only fairly recently, to pose an **affine** version of our **question**: *If* A *is a* k*-algebra and* $A \subset k[x_1, \cdots, x_n] = k^{[n]}$ *(polynomial ring), is* A *a polynomial ring over* k? Then $A \simeq k^{[d]}$, where d is the Krull dimension of A. Again we may assume $n = d$. To avoid obvious counterexamples additional conditions are now required. What comes to mind readily is: A is *affine* (finitely generated over k), *regular* (Spec$A$ is non- singular, so this is inspired by geometry), and *factorial* (A is a UFD). It is a remarkable fact that we have now identified the obstructions, at least for $d = 2$: the **Affine Castelnuovo Theorem** states that *for k algebraically closed of characteristic* 0 *the above conditions imply $A \simeq k^{[2]}$*. An immediate consequence is the **Cancellation Theorem** (in dimension 2, the name comes from geometry): *If A is k-algebra such that $A^{[1]} \simeq k^{[3]}$, then $A \simeq k^{[2]}$*. These results are due to Miyanishi and Sugie [MS], with a crucial contribution by Fujita [F]. Behind them is a formidable machinery of algebraic geometry based on fundamental work of Itaka [I]. An exposition of this theory, which needless to say has taken on a vigorous life of its own and has found numerous other applications, is a main objective of the book.

There is, of course, an (again elementary) **Affine Lüroth Theorem** for $d = 1$ (we only have to assume that A is normal), but no good conditions are known giving $A \simeq k^{[d]}$ if $d \geq 3$. In particular, the cancellation question is wide open for $d \geq 3$. (The answer is negative for the original Zariski cancellation problem for fields [BC-TSS-D].)

In the geometric reformulation of the question (we restrict ourselves to algebraically closed k and $d = 2$) we find a non-singular and complete (compact if $k = \mathbf{C}$) algebraic surface X with function field $k(X) = L$. "$L \subset k^{(n)}$" translates into: "*X is birationally dominated by projective space P^n*," and we ask: "*Is X birationally equivalent to P^2?*" The power of algebraic geometry now derives from the study of *linear systems of curves (effective divisors)* on X. If D is a divisor (linear combination of irreducible curves on X, with integer coefficients, all non-negative if D is effective), the *complete linear system $|D|$ of effective divisors linearly equivalent to D* is a finite dimensional projective space $P^{\ell(D)}$. (D' is linearly equivalent to D if $D' - D$ is the divisor of a rational function on X.) Differential calculus gives us the *canonical divisor class*, the divisors K_X of rational 2-differentials on X. The powerful Riemann-Roch theorem allows to compute, or at least efficiently estimate, $\ell(D)$. (This involves K_X and the intersection product defined on $\mathrm{Pic} X$, the group of divisors modulo linear equivalence.)

To D there is associated a rational map $\varphi_D : X \to P^{\ell(D)}$. Put $\kappa(D) = \sup\{\dim \varphi_{nD}(X) | n > 0\}$. (Formally, $\kappa(D) = -\infty$ if $|nD| = \emptyset$ for $n > 0$, and $0 \leq \kappa(D) \leq 2$ otherwise.) The *Kodaira dimension* of X now is

$$\kappa(X) = \kappa(K_X).$$

It depends on X only up to birational equivalence and $\kappa(X) \leq \kappa(Y)$ if Y dominates X (separably). A fundamental result gives: $\kappa(X) = -\infty$ *if and only if X is birationally ruled, i.e., birationally equivalent to $C \times P^1$, C a curve.* This **Ruledness Theorem** (the if part is easy, the other not) is the beginning of the Enriques-Kodaira classification of surfaces. The Castelnuovo Theorem is an easy consequence.

In the situation of the Affine Castelnuovo Theorem we are geometrically dealing with an open (non-complete) non-singular surface X. ($X = \mathrm{Spec}\, A$, and X is dominated by affine space$\mathbf{A}^n$.) It is possible to embed X as an open subset in a complete, non-singular surface $\overline{X}$ so that the *boundary*

$$\overline{X} \backslash X = D = \sum C_i$$

is a *divisor with simple normal crossing* (SNC-divisor, meaning that the C_i are non-singular, at most two meeting at a point, and if so normally). This statement requires resolution of singularities for surfaces and embedded resolution for curves in a surface. A key insight of Itaka [I] gives: $\kappa(K_{\overline{X}} + D)$ *depends on* X *only, as long as* SNC *holds for* D. (This is an instance of a more general phenomenon, even in higher dimensions. One introduces the sheaf $\Omega_{\overline{X}}(D)$ of rational differential one-forms with simple poles along D. Performing standard linear algebra constructions on $\Omega_{\overline{X}}(D)$ and taking global sections one obtains finite dimensional vectorspaces that are invariants of X.) We are thus justified to introduce

$$\overline{\kappa}(X) = \kappa(K_{\overline{X}} + D)$$

as the (*logarithmic*) *Kodaira dimension of* X.

With the exception of $\overline{X} = P^2$, a complete, non-singular surface is birationally ruled if and only if $\overline{X}$ contains an open subset of the form $C' \times P^1$, C' a curve. We define: a non-singular surface X is *affine-ruled* if X contains an open subset of the form $C' \times \mathbf{A}^1$, C' a curve. That this concept, introduced in [MS], gives a good notion of ruledness in the open context is attested to by the **Affine-Ruledness Theorem** [MS] [F], one of the major results of the book: *Let* X *be a non-singular surface over* k. *Suppose* X *is connected at infinity and* $\overline{\kappa}(X) = -\infty$. *Then* X *is affine-ruled.* (X is connected at infinity if $\overline{X} \backslash X$ is connected. This is the case if X is affine.) An important consequence is the **Characterization Theorem** (for the affine plane) [MS] [F]: *Let* $X =$ *Spec* A *be an affine non-singular surface over* k *with* A *factorial,* $A^* = k^*$ *and* $\overline{\kappa}(X) = -\infty$. *Then* $X \simeq \mathbf{A}^2$. ($A^* =$ group of units of A). The Affine Castelnuovo Theorem and the Cancellation Theorem are immediate consequences. In case $k = \mathbf{C}$ the condition "A is factorial and $A^* = k^*$" could be replaced by "$H_i(X, Z) = 0$ for $i > 0$". Finding conditions that characterize $\mathbf{A}^d$, $d \geq 3$, on the other hand, is an outstanding (in every sense) challenge.

Here is a brief rundown of the contents of the book.

Chapter 1 gives a rapid introduction to complete algebraic surfaces, culminating in the Enriques-Kodaira classification. Many of the basic results are not proven but precise references are given for most. Among not so standard topics there is an exposé of Mori's theory (of extremal rays in the effective cone) in the case of surfaces, and a description of rational, in particular quotient, surface singularities and their resolutions.

Chapter 2 turns to open surfaces. Their Kodaira dimension is introduced. The *Zariski-Fujita decomposition of a pseudo-effective divisor* into its numerically effective (nef) and negative parts is presented in detail. This generalization by Fujita [F] of a result of Zariski is a crucial technical tool in the "open" context. A proof of the Affine-Ruledness Theorem then follows.

As for complete surfaces, there is a notion of *relatively minimal model* in the open case. The *theory of peeling*, described next, is a concrete way to construct such a model. The negative part of $K_{\overline{X}} + D$ is computed. It is supported, it turns out, on certain pieces of D that are resolution graphs of quotient singularities. Contracting these to normal points, we land in the realm of *log projective surfaces*, which are treated next. The tools are then at hand to establish structure theorems for open surfaces according to Kodaira dimension (due to Iitaka, Fujita, Kawamata, Miyanishi, Tsunoda,.....). Let me mention specifically the large effort put into generalizing the Affine-Ruledness Theorem. (This includes a section on *log del Pezzo surfaces*. Mori theory is appealed to.) Here is a somewhat condensed version [MT] (with $k = \mathbf{C}$): *If* $\overline{\kappa}(X) = -\infty$ *and at least one connected component of* D *is not contractible to a normal point, then* X *is affine-ruled or* X *contains an open subset (with good control over the complement) of the form* $\mathbf{C}^2/G\backslash\{0\}$, G *a non- abelian small finite subgroup of* $GL_2(\mathbf{C})$. This theorem applies to the smooth locus of a normal affine surface and is a fundamental structural result in that context. It is, for instance, an essential ingredient in the linearization of $\mathbf{C}^*$-actions on $\mathbf{C}^3$ [KKM-LR].

Chapter 3 reaps rewards for previous hard work with applications to affine surfaces, among them: $\mathbf{A}^1_*$-fibrations (the open analogue of elliptic fibrations), the Characterization and Cancellation Theorems, normal surfaces X with a finite map $\mathbf{A}^2 \to X$, proofs of the theorems of Abhyankar-Moh-Suzuki (*a curve in* $\mathbf{C}^2$ *isomorphic to* $\mathbf{C}$ *is a straight line up to automorphism*) and Lin-Zaidenberg (*a topologically contractible curve in* $\mathbf{C}^2$ *is the zero locus of* $x^p - y^q$ *in suitable coordinates*) with open surface techniques. The book closes with the classification theory of *homology* and *Q-homology planes* (surfaces with $H_i(X, Z) = 0$, resp. $H_i(X,Q) = 0$, for $i > 0$).

In a way, the concept *open variety = completion + extra baggage of the (made nice) boundary divisor* fits seamlessly into the framework of classical algebraic geometry. The baggage can be heavy, but carrying it along brings dividends. In dimension 2, a mature theory of open surfaces has developed in parallel to the classical case. Fascinating on its own, it also has powerful applications to what is emerging as "affine algebraic geometry", the study of affine spaces and closely related varieties. The book under review is an authoritative and comprehensive account of this theory (by one of its creators). It does not make for an easy read, but this may well be inherent in the subject. (Some familiarity with algebraic geometry is recommended to the reader.) It will be an indispensable reference work for experts in the field. I expect it will do a lot to open this exciting area of mathematics to newcomers.

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