Although everyone has an intuitive perception of the concept of “order”, it is by no means obvious how to define and to characterize order in mathematical terms. As often, it is a good idea to look what Nature has to offer. In the physical world, the paradigm of order is a perfect crystal. Its regular macroscopic features derive from an underlying lattice structure where the atoms are arranged in a periodic fashion. The lattice structure can also be probed directly by diffraction experiments, for instance with X-rays or electrons. The diffraction images consist of beautiful point patterns, manifesting the long-range order and the symmetry of the atomic positions.

Some twenty years ago, a surprising discovery shook the perfect world of crystals. Certain materials, so-called quasicrystals, show diffraction patterns that are as point-like as those of crystals, but with symmetries that are incompatible with lattice periodicity. Their mathematical description requires more general ordered structures than lattices. This stimulated the exploration of the vast expanse of possible manifestations of order that lie between the extremes
of lattice periodicity and random disorder. There are still many open questions regarding such structures. To give an example, it is not known whether one can find a single tile such that the entire plane can be tiled by copies of it, but that any such tiling is aperiodic.

This is the fascinating realm of aperiodic order that is addressed by the contributions collected in the book under review. The title “Directions in Mathematical Quasicrystals” may be taken literally — the mathematics of aperiodic order is a young research topic, and this book is the third that arose from the research activity initiated by a workshop at the Fields Institute in 1995. As the two previous volumes (The mathematics of long-range aperiodic order, edited by R. V. Moody, Kluwer, Dordrecht, 1997; Quasicrystals and Discrete Geometry, edited by J. Patera, American Mathematical Society, Providence RI, 1998), it contains articles that explore various topics related to aperiodic order, emphasizing the deep and sometimes astonishing connections to many areas of mathematics. In particular, this includes algebraic number theory, combinatorics, geometry, measure theory, dynamical systems, \(C^*\)-algebras and \(K\)-theory, which all yield different views of the central topic, aperiodically ordered point sets and tilings in Euclidean space. These present different directions for the construction and the characterization of such sets, the topic as such not yet being complete in the sense that a definitive characterization of a “mathematical quasicrystal” still has to be found. But this is not the aim of this volume, it serves to make the picture more coherent by putting together several pieces in the puzzle of aperiodic order, and by discussing how the various approaches used to construct aperiodically ordered systems are related to each other.

The individual articles were specifically written for this volume, by leading scientists in the field, among them several Canadian mathematicians. The articles present an overview of the current knowledge and also address many of the open questions. Some contributions are of introductory character and are easily accessible for readers who are not familiar with the topic, while some more technical papers contain detailed proofs of non-trivial results. Thus the mathematical background knowledge expected of the reader varies quite a bit between the contributions, but this is compensated by carefully prepared references to the literature.

A major part of the book is devoted to the characterization of point sets and tilings, in particular in terms of diffraction, which in mathematical terms is the Fourier transform of the autocorrelation measure. The construction of topological invariants for tiling spaces is beautifully explained, paving the way to a second main topic of the book, the spectral theory of aperiodic Schrödinger operators. The table of contents comprises the following entries: “Self-similar measures for quasicrystals” (M. Baake, R. V. Moody), “Fourier analysis of deformed model sets” (G. Bermann, M. Duneau), “Mathematical quasicrystals and the problem of diffraction” (J. C. Lagarias), “Designer quasicrystals: Cut-and-project sets with pre-assigned properties” (P. A. B. Pleasants), “Generalized model sets and dynamical systems” (M. Schlottmann), “On shelling icosahedral quasicrystals” (A. Weiss), “Tilings, \(C^*\)-algebras, and \(K\)-theory” (J. Kellendonk,

This collection provides ideal reference material for researchers who are active in the field as well as for any mathematician or theoretical physicist who is interested to learn more about this fascinating topic. It gives an up-to-date account of the present state of knowledge and monitors the rapid evolution of this intriguing field.