Every complex matrix is similar to a matrix in upper triangular form. This immediately exhibits certain information such as the eigenvalues (including multiplicity) and a collection of invariant subspaces. While the triangular form does not contain the detailed algebraic information contained in the Jordan form, it has several advantages—for example, it is more computable and it can be obtained by a unitary similarity. It is the connection with invariant subspaces which attracted the authors. The focus of this nice little book is on the development of the upper triangular form in two directions. First they wish to find triangular form theorems for larger classes of matrices with a particular focus on semigroups, and secondly they discuss infinite dimensional analogues on Hilbert and Banach spaces.

In finite dimensions there are many classical and beautiful results about simultaneous triangularization that do not find their way into any linear algebra text that I know of. Here is an example. Notice that if $\mathcal{A}$ is an algebra of matrices which is triangularizable, then every commutator $AB - BA$ for $A, B \in \mathcal{A}$ is strictly upper triangular and hence nilpotent. The converse turns out to be true,
but perhaps not very checkable. McCoy’s Theorem provides a much stronger converse when the algebra $A$ is generated by two matrices $A$ and $B$. If each matrix $p(A, B)(AB - BA)$ is nilpotent for every non-commuting polynomial in $A$ and $B$, then $A$ and $B$ are simultaneously triangularizable and thus so is the algebra that they generate.

Various conditions such as commutativity allow simultaneous triangularization. Two other classical results are Engel’s Theorem: every Lie algebra of nilpotent matrices is triangularizable, and Levitzki’s Theorem: every semigroup of nilpotent matrices is triangularizable. All of this material is contained in the first forty pages. It is nicely written, and could be used for an undergraduate seminar.

The next two chapters deal with various conditions on semigroups that imply or are related to triangularizability. I will mention one nice result. Say that the spectrum is sublinear on a set $S$ if $\sigma(S + \lambda T) \subset \sigma(S) + \lambda \sigma(T)$ for every $S, T \in S$. Then a semigroup $S$ is triangularizable if and only if the spectrum is sublinear on $S$. I must admit that I found these chapters too encyclopedic, so I skipped ahead to the chapters on operator theory.

Chapter 6 is an introduction to compact operators on Banach space. It could be read by a sophisticated student who has not had functional analysis, but I suspect that most readers of the last half of this book would be familiar with these ideas. They quickly get to invariant subspace results, beginning with the famous result of Lomonosov: if $K$ is a non-zero compact operator, then there is a proper subspace which is invariant for the commutant of $K$. In infinite dimensions, a triangular form does not always mean a discrete basis in which the operator in upper triangular. One has to expand the notion to allow a maximal chain of closed subspaces (a nest) which are invariant. For example the Volterra operator on $L^p(0, 1)$ given by $Vf(x) = \int_0^x f(t) \, dt$ has invariant subspaces $L^p(t, 1)$ for every $t \in [0, 1]$, and there are no others. This provides a continuous triangularization for $V$.

Chapter 7 contains a treatment of Ringrose’s beautiful theorem that every compact operator $K$ has a (generalized) upper triangular form, and the non-zero points in the spectrum (including multiplicity) can be read off as the diagonal entries corresponding to discrete parts of the nest—that is, if $\mathcal{M}_- \subset \mathcal{M}$ are two elements of the nest with $\dim(\mathcal{M}/\mathcal{M}_-) = 1$, then there is a unique scalar $\lambda$ such that $(K - \lambda I)\mathcal{M} \subset \mathcal{M}_-$ and $\lambda$ is an eigenvalue for $K$. In the Hilbert space case, this is more transparent. There is a unit vector $e \in \mathcal{M} \oplus \mathcal{M}_-$ and $\lambda = \langle Ke, e \rangle$. This theorem should be better known, and could easily be included in an introductory course in functional analysis or operator theory.

Returning to the question of triangularizability, there is the analogue of McCoy’s Theorem for compact operators replacing the word nilpotent by quasinilpotent, which means spectrum $\{0\}$. Chapter 8 deals with semigroups of compact operators in detail. The highlight is a recent theorem of Turovskii: every semigroup of compact quasinilpotent operators on a Banach space is triangularizable. The analogue of the result on the equivalence of triangularizability and sublinearity of the spectrum goes through for semigroups of compact operators.

The last chapter deals with bounded operators. Here the area consists mainly
of open questions and interesting counterexamples. For example, there is a semi-
group of nilpotent operators of index 2 on a Hilbert space with no invariant sub-
spaces! Naturally one serious barrier is the famous Invariant Subspace Problem: 

\textit{does every bounded operator on Hilbert space have a proper invariant subspace?}

The situation is worse in Banach space, where there are counterexamples. How-
ever there are some sorts of hypotheses which lead to triangularizability even here.

Overall this is a very nicely written book. For the student or expert in 
operator theory, it is a stimulating read. For the dilettante, I would suggest 
the first two chapters for the matrix theory, and then the first few sections of 
chapters 7, 8 and 9 for a taste of the infinite dimensional theory.