Problems

1. Find all integer solutions to the equation \(7x^2y^2 + 4x^2 = 77y^2 + 1260\).

2. A polynomial \(f(x)\) with integer coefficients is said to be tri-divisible if \(3\) divides \(f(k)\) for any integer \(k\). Determine necessary and sufficient conditions for a polynomial to be tri-divisible.

3. Let \(N\) be a 3-digit number with three distinct non-zero digits. We say that \(N\) is mediocre if it has the property that when all six 3-digit permutations of \(N\) are written down, the average is \(N\). For example, \(N = 481\) is mediocre, since it is the average of \(\{418, 481, 148, 184, 814, 841\}\). Determine the largest mediocre number.

4. Given an acute-angled triangle \(ABC\) whose altitudes from \(B\) and \(C\) intersect at \(H\), let \(P\) be any point on side \(BC\) and \(X, Y\) be points on \(AB, AC\), respectively, such that \(PB = PX\) and \(PC = PY\). Prove that the points \(A, H, X, Y\) lie on a common circle.

5. Let \(x\) and \(y\) be positive real numbers such that \(x + y = 1\). Show that
\[
\left(\frac{x + 1}{x}\right)^2 + \left(\frac{y + 1}{y}\right)^2 \geq 18.
\]

6. Let \(\triangle ABC\) be a right-angled triangle with \(\angle A = 90^\circ\), and \(AB < AC\). Let points \(D, E, F\) be located on side \(BC\) so that \(AD\) is the altitude, \(AE\) is the internal angle bisector, and \(AF\) is the median.
Prove that \(3AD + AF > 4AE\).

7. A \((0_x, 1_y, 2_z)\)-string is an infinite ternary string such that:
   - If there is a 0 in position \(i\), then there is a 1 in position \(i + x\)
   - If there is a 1 in position \(j\) then there is a 2 in position \(j + y\)
   - if there is a 2 in position \(k\) then there is a 0 in position \(k + z\).

For how many ordered triples of positive integers \((x, y, z)\) with \(x, y, z \leq 100\) does there exist \((0_x, 1_y, 2_z)\)-string?

8. A magical castle has \(n\) identical rooms, each of which contains \(k\) doors arranged in a line. In room \(i\), \(1 \leq i \leq n - 1\) there is one door that will take you to room \(i + 1\), and in room \(n\) there is one door that takes you out of the castle. All other doors take you back to room 1. When you go through a door and enter a room, you are unable to tell what room you are entering and you are unable to see which doors you have gone through before. You begin by standing in room 1 and know the values of \(n\) and \(k\). Determine for which values of \(n\) and \(k\) there exists a strategy that is guaranteed to get you out of the castle and explain the strategy. For such values of \(n\) and \(k\), exhibit such a strategy and prove that it will work.