1. Let \( a_1, a_2, \ldots, a_n \) be positive real numbers whose product is 1. Show that the sum
\[
\frac{a_1}{1 + a_1} + \frac{a_2}{(1 + a_1)(1 + a_2)} + \frac{a_3}{(1 + a_1)(1 + a_2)(1 + a_3)} + \cdots + \frac{a_n}{(1 + a_1)(1 + a_2) \cdots (1 + a_n)}
\]
is greater than or equal to \( \frac{2^n - 1}{2^n} \).

2. Let \( m \) and \( n \) be odd positive integers. Each square of an \( m \) by \( n \) board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of \( m \) and \( n \).

3. Let \( p \) be a fixed odd prime. A \( p \)-tuple \( (a_1, a_2, a_3, \ldots, a_p) \) of integers is said to be good if
   (i) \( 0 \leq a_i \leq p - 1 \) for all \( i \), and
   (ii) \( a_1 + a_2 + a_3 + \cdots + a_p \) is not divisible by \( p \), and
   (iii) \( a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_pa_1 \) is divisible by \( p \).
Determine the number of good \( p \)-tuples.

4. The quadrilateral \( ABCD \) is inscribed in a circle. The point \( P \) lies in the interior of \( ABCD \), and \( \angle PAB = \angle PBC = \angle PCD = \angle PDA \). The lines \( AD \) and \( BC \) meet at \( Q \), and the lines \( AB \) and \( CD \) meet at \( R \). Prove that the lines \( PQ \) and \( PR \) form the same angle as the diagonals of \( ABCD \).

5. Fix positive integers \( n \) and \( k \geq 2 \). A list of \( n \) integers is written in a row on a blackboard. You can choose a contiguous block of integers, and I will either add 1 to all of them or subtract 1 from all of them. You can repeat this step as often as you like, possibly adapting your selections based on what I do. Prove that after a finite number of steps, you can reach a state where at least \( n - k + 2 \) of the numbers on the blackboard are all simultaneously divisible by \( k \).